Further Analysis

Prof. W.T. Gowers

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Introduction

These notes are based on the course "Further Analysis" given by Prof. W.T. Gowers[1] in Cambridge in the Lent Term 1997. These typeset notes are totally unconnected with Prof. Gowers.

Other sets of notes are available for different courses. At the time of typing these courses were:

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General Relativity	Dynamical Systems
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[1] Yes, *that* Prof. Gowers.

Chapter 1

Topological Spaces

1.1 Introduction

Definition 1.1. *A topological space is a set X together with a collection τ of subsets of X satisfying the following axioms.*

1. $\emptyset, X \in \tau$;
2. *If* $U_1, \ldots, U_n \in \tau$, *then* $U_1 \cap \cdots \cap U_n \in \tau$; *(that is, τ is closed under finite intersections)*
3. *Any union of sets in τ is in τ (or τ is closed under any unions).*

Definition 1.2. *τ is called a topology on X. The sets in τ are called open sets. A subset of X is closed if its complement is open.*

Examples.

1. *If (X, d) is a metric space and τ the collection of open sets (in a metric space sense) then (X, τ) is a topological space.*
2. *If X is any set and τ is the power set of X, (X, τ) is a topological space. τ is called the discrete topology on X.*
3. *If X is any set and $\tau = \{\emptyset, X\}$, (X, τ) is a topological space. τ is called the indiscrete topology on X.*
4. *If X is any infinite set, and $\tau = \{Y \subset X : X \setminus Y \text{ is finite}\} \cup \{\emptyset\}$ then (X, τ) is a topological space. τ is called the cofinite topology on X.*
5. *If X is any uncountable set, and $\tau = \{Y \subset X : X \setminus Y \text{ is countable}\} \cup \{\emptyset\}$ then (X, τ) is a topological space. τ is called the cocountable topology on X.*

Definition 1.3. *Let A be a subset of a topological space. The closure of A, denoted $\overline{A}$, is the intersection of all closed sets containing A. Note that $\overline{A}$ is closed and any closed set containing A contains $\overline{A}$.*

Definition 1.4. *Let A be a subset of a topological space. The interior of A, denoted $\operatorname{int} A$ or A^0 is the union of all open sets in A. Note that $\operatorname{int} A$ is open and any open set in A is in $\operatorname{int} A$.*

Definition 1.5. *The boundary ∂A of a set A is $\overline{A} \setminus \operatorname{int} A$.*

Definition 1.6. *Let $(x_n)_1^\infty$ be a sequence in a topological space (X, τ). We say that x_n converges to x ($x_n \to x$) if for every open set U such that $x \in U$, $\exists\, N$ such that $n \geq N \Rightarrow x_n \in U$. This agrees with the usual definition for metric spaces.*

Definition 1.7. *Let (X, τ) and (Y, σ) be topological spaces and let $f\colon X \mapsto Y$. We say that f is continuous if for every $U \in \sigma$, $f^{-1}(U) \in \tau$. (Or inverse image of an open set is open.)*

It follows from results in Analysis that this definition agrees with the usual $\epsilon - \delta$ definition if X and Y are metric spaces.

Definition 1.8. *Let (X, τ) be a topological space and let $x \in X$. A neighbourhood of x is a set N that contains an open set containing x.*

Proposition 1.9. *Let (X, τ) and (Y, σ) be topological spaces and let $f\colon X \mapsto Y$. Then the following are equivalent :-*

1. *f is continuous.*
2. *For every $x \in X$ and every neighbourhood M of $f(x)$ there exists a neighbourhood N of x such that $f(N) \subset M$.*

Proof. Firstly do $1 \Rightarrow 2$. M contains an open set U containing $f(x)$. Then $N = f^{-1}(U)$ is a neighbourhood of x such that $f(N) \subset U$.

Now do the case $2 \Rightarrow 1$. Let U be an open subset of Y. For every $x \in f^{-1}(U)$, U is a neighbourhood of $f(x)$. We can find a neighbourhood N_x of x such that $f(N_x) \subset U$. Now N_x contains an open set V_x containing x. Let

$$V = \bigcup_{x \in f^{-1}(U)} V_x.$$

Then $f(V_x) \subset U\ \forall x$, so $f(V) \subset U$ and $V \subset f^{-1}(U)$. But $V \supset f^{-1}(U)$ so $V = f^{-1}(U)$. V is open, so $f^{-1}(U)$ is open and f is continuous. □

Definition 1.10. *Let (X, τ) and (Y, σ) be topological spaces and $f\colon X \mapsto Y$. f is a homeomorphism if it is continuous with a continuous inverse.*

Example. *$\mathbb{R}$ and $(0, 1)$ with usual metrics are homeomorphic.*

1.2 Building New Spaces

Definition 1.11. *Let (X, τ) be a topological space and $Y \subset X$. The subspace topology on Y is*

$$\{U \cap Y : U \in \tau\}.$$

Proposition 1.12. *Let (X, d) be a metric space, $Y \subset X$ and $U \subset Y$. Then the following are equivalent.*

1. *U is open in the subspace topology on Y.*
2. *For every $u \in U, \exists\, \delta > 0$ such that if $v \in Y, d(u, v) < \delta$ then $v \in U$.*

Proof. Do $1 \Rightarrow 2$. Let $u \in U$. Since U open in Y, $\exists\ V \subset X$, V open such that $U = V \cap Y$. Then $\exists\ \delta > 0$ such that $d(u, v) < \delta \Rightarrow v \in V$. Now $d(u, v) < \delta$ and $v \in Y \Rightarrow v \in V \cap Y = U$.

Now do $2 \Rightarrow 1$. Suppose U satisfies 2. For every $u \in U$, pick $\delta = \delta(u) > 0$ such that $d(u, v) < \delta$ and $v \in Y \Rightarrow v \in U$. Let $N_u = \{v \in X : d(u, v) < \delta\}$. Now N_u is open and $V = \bigcup_{u \in U} N_u$ is open. Now $V \cap Y = U$. ☐

Definition 1.13. *Let (X, τ) be a topological space and $\sim$ be an equivalence relation on X. Denote the set $X/\!\sim$ by Y and let $q \colon X \mapsto Y$ be the equivalence map. (ie if $x \in X$, $q(x)$ is the equivalence class of x). The quotient topology on Y is $\{U \in Y : q^{-1}(U) \in \tau\}$.*

N.B.

1. *$q^{-1}(U)$ is the union of the equivalence classes in U.*
2. *Notice q is continuous and that the quotient topology is the largest topology making it so.*

Examples.

1. *Let τ be the usual topology on $\mathbb{R}$. Then the subspace topology on $\mathbb{Z} \subset \mathbb{R}$ coincides with the discrete topology on $\mathbb{Z}$.*
2. *Let τ be as in 1. Then the interval $(\frac{1}{2}, 1]$ is open in the subspace topology on $[0, 1]$.*
3. *Let $\sim$ be the equivalence relation on $\mathbb{R}$ defined by $x \sim y \Leftrightarrow x - y \in \mathbb{Z}$. Let $[x]$ denote the equivalence class of x. Then the map*

$$\phi \colon \mathbb{R}/\!\sim\ \mapsto \mathbb{T} \equiv \{z \in \mathbb{C} : |z| = 1\}$$

is both well defined and a homeomorphism.

Definition 1.14. *Let (X, τ) and (Y, σ) be topological spaces. The product topology on $X \times Y$ is the collection of all possible unions of sets of the form $U \times V$ with $U \in \tau$ and $V \in \sigma$. Similarly, if $(X_i, \tau_i)_{i=1}^n$ are n topological spaces, the product topology on $\prod_{i=1}^n X_i$ is the collection of all unions of sets $\prod_{i=1}^n U_i$ with $U_i \in \tau_i$.*

Example. *If $\mathbb{R}$ has its usual topology, then the product topology on $\mathbb{R} \times \mathbb{R}$ is the same as the usual topology on $\mathbb{R} \times \mathbb{R} \equiv \mathbb{R}^2$.*

Proof. Let us write π for the product topology on $\mathbb{R} \times \mathbb{R}$ and σ for the usual (Euclidean) topology. Let $U \in \sigma$. Then given $(x_1, x_2) \in U$, $\exists\ \delta > 0$ such that $d((x_1, x_2), (y_1, y_2)) < \delta \Rightarrow (y_1, y_2) \in U$. Then the π-open set $(x - \frac{\delta}{2}, x + \frac{\delta}{2}) \times (x - \frac{\delta}{2}, x + \frac{\delta}{2}) \subset U$ and contains (x_1, x_2), so U is π-open, given $\sigma \subset \pi$. Conversely, every set of the form $A \times B$ with A, B open in $\mathbb{R}$ is open in $\mathbb{R}^2$. The union of such sets is σ-open, giving $\pi \subset \sigma$ and $\pi = \sigma$. ☐

Exercise to show that $X \times (Y \times Z) = X \times Y \times Z$ as topological spaces.

Definition 1.15. *Let (X, τ) be a topological space. A basis for τ (or a basis of open sets) is a subset $\beta \subset \tau$ such that every $U \in \tau$ is a union of sets in β. The sets in β are called basic open sets. If $x \in X$, then a basis of neighbourhoods of x is a collection $\mathcal{N}$ of neighbourhoods of x such that every neighbourhood of x contains $N \in \mathcal{N}$.*

Examples.

The sets $U \times V$ with $U \in \tau$ and $V \in \sigma$ are a basis for the product topology on $(X, \tau) \times (Y, \sigma)$.

The sets $\{y : d(x, y) \leq \frac{1}{n}\}$ *are a basis of neighbourhoods for a point* x *in a metric space.*

Proposition 1.16. *The quotient topology and product topology are topologies.*

Proof. The result for the quotient topology follows easily from the fact that q^{-1} preserves unions and intersections. For products let $(X_i, \tau_u)_{i=1}^n$ be topological spaces. Everything is simple except closure under finite intersections. By induction it is sufficient to prove for two sets.

If $U_i \in \tau_i$, call $U_1 \times \cdots \times U_n$ an open box[1]. Firstly, observe that

$$(U_1 \times \cdots \times U_n) \cap (V_1 \times \cdots \times V_n) = (U_1 \cap V_1) \times \cdots \times (U_n \cap V_n)$$

and thus the intersection of two open boxes is an open box. Now take

$$\bigcup_{\gamma \in \Gamma} B_\gamma \text{ and } \bigcup_{\delta \in \Delta} C_\delta,$$

with all B_γ's and C_δ's open boxes.

Now

$$\bigcup_{\gamma \in \Gamma} B_\gamma \cap \bigcup_{\delta \in \Delta} C_\delta = \bigcup_{\substack{\gamma \in \Gamma \\ \delta \in \Delta}} B_\gamma \cap C_\delta,$$

which is a union of open boxes, and so open. □

[1] This is not standard terminology.

Chapter 2

Compactness

2.1 Introduction

Definition 2.1. *Let (X, τ) be a topological space. An open cover of X is a collection $\{U_\gamma : \gamma \in \Gamma\}$ of open sets such that $X = \bigcup_{\gamma \in \Gamma} U_\gamma$. If $Y \subset X$ then an open cover of Y is a collection $\{U_\gamma : \gamma \in \Gamma\}$ such that $Y \subset \bigcup_{\gamma \in \Gamma} U_\gamma$.*

A subcover of a cover $\mathcal{U} = \{U_\gamma : \gamma \in \Gamma\}$ is a subset $\mathcal{V} \subset \mathcal{U}$ which is still an open cover.

Examples.

1. *$\{I_n = (-n, n), n = 1, 2, \ldots\}$ is an open cover for $\mathbb{R}$. $\{I_{n^2}\}$ is a subcover.*

2. *The intervals $I_n = (n-1, n+1)$ with $n \in \mathbb{Z}$ form a cover of the reals with no proper subcover.*

Definition 2.2. *A topological space (X, τ) is compact if every open cover has a finite subcover.*

Examples. *The open covers mentioned above show that $\mathbb{R}$ is not compact. Any finite topological space is compact, as is any set with the indiscrete topology.*

2.2 Some compact sets

Lemma 2.3. *Let (X, τ) be a topological space with $Y \subset X$. Then the following are equivalent :-*

1. *Y is compact in the subspace topology.*

2. *Every cover of Y by $U_\gamma \in \tau$ has a finite subcover.*

Proof. $1 \Rightarrow 2$. Let $Y \subset \bigcup_{\gamma \in \Gamma} U_\gamma$ with $U_\gamma \in \tau$. Then $Y = \bigcup_{\gamma \in \Gamma}(U_\gamma \cap Y)$ with U_γ open in Y. Since Y is compact $\exists\, \gamma_1, \ldots, \gamma_n$ such that $Y = \bigcup_{i=1}^n (U_{\gamma_i} \cap Y)$, which gives that $Y \subset \bigcup_{i=1}^n U_{\gamma_i}$.

$2 \Rightarrow 1$. Let $Y = \bigcup_{\gamma \in \Gamma} V_\gamma$ with V_γ open in Y. Since $V_\gamma = U_\gamma \cap Y$ for some U_γ open in X, $Y \subset \bigcup_{\gamma \in \Gamma} U_\gamma$. By assumption, $\exists\, \gamma_1, \ldots, \gamma_n$ such that $Y \subset \bigcup_{i=1}^n U_{\gamma_i}$. This implies that $Y = \bigcup_{i=1}^n (U_{\gamma_i} \cap Y) = \bigcup_{i=1}^n V_{\gamma_i}$. □

Theorem 2.4 (The Heine-Borel Theorem). *The closed interval* $[a, b] \subset \mathbb{R}$ *is compact.*

Proof. Let $[a, b] \subset \bigcup_{\gamma \in \Gamma} U_\gamma$ with U_γ open in $\mathbb{R}$ and let

$$K = \{x \in [a, b] : [a, x] \text{ is contained in a finite union of open sets}\}.$$

Let $r = \sup K$. Then $r \in [a, b]$, so $r \in U_\gamma$ for some γ. But U_γ is open, so $\exists\ \delta > 0$ such that $[r - \delta, r + \delta] \subset U_\gamma$. By the definition of r, $[a, r - \delta]$ has a finite open cover, thus $[a, r + \delta]$ has a finite open cover. This is a contradiction, unless $r = b$. □

Theorem 2.5. *A continuous image of a compact set is compact.*

Proof. Let X be compact and let $f : X \mapsto Y$ be continuous. Now $f(X) \subset \bigcup_{\gamma \in \Gamma} U_\gamma$ with U_γ open in Y. Since f is continuous, $f^{-1}(U_\gamma)$ is open in $X\ \forall\ \gamma \in \Gamma$ and furthermore $X = \bigcup_{\gamma \in \Gamma} f^{-1}(U_\gamma)$. Since X is compact, $\exists\ \gamma_1, \ldots, \gamma_n$ such that $X = \bigcup_{i=1}^n f^{-1}(U_{\gamma_i})$. Thus $f(X) \subset \bigcup_{i=1}^n U_{\gamma_i}$. □

Theorem 2.6. *A closed subset of a compact set is compact.*

Proof. Let X be compact and $K \subset X$ be closed. Let $K \subset \bigcup_{\gamma \in \Gamma} U_\gamma$ with U_γ open in X. Now $X = (X \setminus K) \cup \left(\bigcup_{\gamma \in \Gamma} U_\gamma\right)$. Since X is compact, $\exists$ a finite subcover and $X = (X \setminus K) \cup (\bigcup_{i=1}^n U_{\gamma_i})$. Then $K \subset \bigcup_{i=1}^n U_{\gamma_i}$. □

Definition 2.7. *A topological space* (X, τ) *is called Hausdorff, if given any two distinct* $x, y \in X$*, there exist disjoint open sets* U *and* V *with* $x \in U$ *and* $y \in V$.

Examples. *1. Any metric space is Hausdorff. Given* x, y *distinct, let* $U = \{z : d(x, z) < d(x, y)/2\}$ *and* $V = \{z : d(y, z) < d(x, y)/2\}$.

2. Any set with more than one element and the indiscrete topology is not Hausdorff.

Theorem 2.8. *Every compact subset of a Hausdorff topological space is closed.*

Proof. Let X be Hausdorff and $K \subset X$ be compact. If $x \notin K$ and $y \in K$ then one can find disjoint open sets U_{xy} and V_{xy} with $x \in U_{xy}$ and $y \in V_{xy}$. For fixed x, the sets $V_{xy}, y \in K$ form an open cover of K. Hence $\exists\ y_1, \ldots, y_n$ such that $K \subset \bigcup_{i=1}^n V_{xy_i}$ Let $U_x = \bigcap_{i=1}^n U_{xy_i}$ and $V_x = \bigcup_{i=1}^n V_{xy_i}$. Note that $U_x \cap V_x = \emptyset$, $x \in U_x$, $K \subset V_x$ and $U_x \cap K = \emptyset$. But $\bigcup_{x \notin K} U_x = X \setminus K$ is open. □

Theorem 2.9. *A product of finitely many compact sets is compact.*

Proof. It is enough to do two sets. The general result follows by induction (and $X \times (Y \times Z) = X \times Y \times Z$).

Let X and Y be compact and let $X \times Y = \bigcup_{\gamma \in \Gamma} U_\gamma$, U_γ open in $X \times Y$. U_γ is the union of sets of the form $V \times W$, with V open in X and W open in Y. It follows that $X \times Y = \bigcup_{\delta \in \Delta} V_\delta \times W_\delta$ with V_δ open in X and W_δ open in Y, $V_\delta \times W_\delta \subset U_\gamma$. Let $x \in X$. Now

$$\{x\} \times Y \subset \bigcup_{\substack{\delta \in \Delta \\ x \in V_\delta}} V_\delta \times W_\delta.$$

Since Y is compact, one can find $\delta_1, \ldots, \delta_n$ such that $Y = \bigcup_{i=1}^n W_{\delta_i}$. Let $V_x = \bigcap_{i=1}^n V_{\delta_i}$. The sets V_x form an open cover of X, thus $\exists\ x_1, \ldots, x_n$ such than $X =$

$\bigcup_{j=1}^m V_{x_j}$. Now $X \times Y = \bigcup_{j=1}^m V_{x_j} \times Y$. But $V_{x_j} \times Y$ has a finite cover of $(V_\delta \times W_\delta)$'s. For each $V_\delta \times W_\delta$, choose U_γ with $V_\delta \times W_\delta \subset U_\gamma$ to obtain a finite open cover of $X \times Y$. □

Theorem 2.10. *A subset of $\mathbb{R}^n$ is compact iff it is closed and bounded.*

Proof. Let $X \subset \mathbb{R}^n$ be closed and bounded. Then $\exists\ M$ such that $d(x,0) \leq M \forall\ x \in X$. In particular, $X \subset [-M,M]^n$. But $[-M,M]$ is compact, so $[-M,M]^n$ is compact. But X is closed, so X is compact.

Conversely, if X is compact then X is closed, since $\mathbb{R}^n$ is Hausdorff. If X is not bounded, then the sets $U_m = \{x \in X : d(x,0) < m\}$ form an open cover with no finite subcover. □

2.3 Consequences of compactness

Theorem 2.11. *A continuous real function on a compact metric space is bounded and attains its bounds.*

Proof. The image of such a function is compact, and so closed and bounded. Closed $\Rightarrow$ the function attains its bounds. □

Theorem 2.12. *Let X be a compact metric space, let Y be a metric space and let $f\colon X \mapsto Y$ be continuous. Then f is uniformly continuous.*

Proof. Let $\epsilon > 0$ be arbitrary. We must show $\exists\ \delta > 0\ \forall\ x, y \in X d(x,y) < \delta \Rightarrow d(f(x), f(y)) < \epsilon$. Since f is continuous, $\forall\ x \in X\ \exists\ \delta_x > 0\ \forall\ y \in X d(x,y) < 2\delta_x \Rightarrow d(f(x), f(y)) < \epsilon/2$.

Now let $U_x = \{y : d(x,y) < \delta_x\}$. Then the U_x form an open cover of X, so $\exists\ x_1, \ldots, x_n$ such that $X = \bigcup_{i=1}^n U_{x_i}$. Let $\delta = \min \delta_{x_i}$.

Let $d(y,z) < \delta$. Since the U_{x_i} form a cover we can find i such that $d(x_i, y) < \delta_{x_i}$ and $d(x_i, z) < \delta_{x_i}$. By the definition of the U_{x_i}, $d(f(y), f(z)) < \epsilon$ by the triangle inequality. □

Lemma 2.13. *Let X be a metric space and let $K \subset X$ be compact and $Y \subset X$ be closed. Then $\exists\ x \in K$ such that $d(x,Y) = \inf\{d(x,w) : x \in K, w \in Y\}$.*

Proof. Define $f\colon K \mapsto \mathbb{R}$ by $f(x) = dist(x,Y)$. This is continuous, and so attains its lower bound. □

In particular, if K and Y are disjoint, then $d(x,Y) > 0$ for every x as Y is closed. Hence $\exists\ \delta > 0$ such that $d(x,Y) \geq \delta\ \forall\ x \in X$.

2.4 Other forms of compactness

Definition 2.14. *X is sequentially compact if every sequence in X has a convergent subsequence.*

Bolzano-Weierstrass is the statement that a closed bounded subset of $\mathbb{R}^n$ is sequentially compact.

Theorem 2.15. *A compact metric space is sequentially compact.*

Proof. Let X be a metric space and let $(x_n)_1^\infty$ be a sequence with no convergent subsequence.

Now, claim that $\forall\ x \in X\ \exists\ \delta > 0$ such that $d(x, x_n) < \delta$ for at most finitely many n. Otherwise $\exists\ x$ such that $d(x, x_n) < m^{-1}$ infinitely many times. Now easy to construct a convergent subsequence.

For each x, pick such a δ and let $U_x = \{y : d(x, y) < \delta\}$. The U_x thus form an open cover with no finite subcover. □

Theorem 2.16. *Let $X \subset \mathbb{R}^n$. Then the following are equivalent.*

1. *X is compact.*
2. *X is sequentially compact.*
3. *X is closed and bounded.*

Proof. We know $3 \Leftrightarrow 1 \Rightarrow 2$ from previous theorems. Thus enough to show that $2 \Rightarrow 3$.

If X is not closed, $\exists\ x \notin X$ such that $\forall\ m\ \exists\ y_m \in X$ such that $d(y_m, x) < n^{-1}$. Then every subsequence of y_n converges in $\mathbb{R}^n$ to x and does not converge in X. If X is not bounded then $\forall\ n\ \exists\ x_n$ such that $d(0, x_n) \geq n$, a sequence with no convergent subsequence. □

Chapter 3

Connectedness

3.1 Introduction

Definition 3.1. *Let X be a topological space. Suppose $U, V \subset X$ such that*

1. *U, V open,*
2. *$U \cap V = \emptyset$,*
3. *$X = U \cup V$,*
4. *both U and V are non-empty.*

Then U, V are said to disconnect X. X is connected if no two subsets disconnect X.

If Y is a subspace of a topological space X, then Y is disconnected in the subspace topology iff $\exists\ U, V$ open in X such that $U \cap Y$, $V \cap Y \neq \emptyset$ and $Y \subset U \cup V$ and $Y \cap U \cap V = \emptyset$. In this case we shall say that U, V disconnect Y.

Proposition 3.2. *Let X be a topological space. Then the following are equivalent.*

1. *X is connected.*
2. *Every continuous $f : X \mapsto \mathbb{Z}$ is constant.*
3. *The only subsets of X both open and closed are $\emptyset$ and X.*

Proof. $1 \Rightarrow 2$. Suppose $\exists$ non-constant $f : X \mapsto \mathbb{Z}$. Then we can find $m < n$ such that both m and n are in $f(X)$. Then $f^{-1}(\{k : k \leq m\})$ and $f^{-1}(\{k : k > m\})$ disconnect X.

$2 \Rightarrow 1$. Suppose U and V disconnect X. Then consider

$$f(x) = \begin{cases} 0 & x \in U \\ 1 & x \in V. \end{cases}$$

$1 \Leftrightarrow 3$. Note that if $U \subset X$, $U \cup (X \setminus U) = X$. □

Proposition 3.3. *A continuous image of a connected space is connected.*

Proof. Let $f : X \mapsto Y$ be continuous with X a connected topological space. Then if U and V disconnect $f(X)$, $f^{-1}(U)$ and $f^{-1}(V)$ disconnect X. □

3.2 Connectedness in $\mathbb{R}$

Definition 3.4. *A subset I of the reals is an interval if, whenever $x \leq y \leq z$, $x, z \in I \Rightarrow y \in I$.*

Every interval is of one of these nine forms: $[a, b]$, $[a, b)$, $[a, \infty)$, $(a, b]$, (a, b), (a, ∞), $(-\infty, b)$, $(-\infty, b]$, $(-\infty, \infty)$.

To see this note that at the upper end of the interval there are three possibilities - bounded and achieves bound, bounded and does not achieve bound and unbounded. Similarly for the lower end of the interval.

Theorem 3.5. *A subset of $\mathbb{R}$ is connected iff it is an interval.*

Proof. If $X \subset \mathbb{R}$ is not an interval we can find $x \leq y \leq z$ such that $x, z \in X$ and $y \notin X$. Then $(-\infty, y)$ and (y, ∞) disconnect X.

Now let $I \subset \mathbb{R}$ be an interval and suppose U and V disconnect I. We can find $u \in U \cap I$ and $v \in V \cap I$ and without loss of generality take $u < v$. Since I is an interval, $[u, v] \subset I$. Let $s = \sup\{[u, v] \cap U\}$. If $s \in U$ then $s < v$. Since U is open $\exists\ \delta > 0$ such that $(s - \delta, s + \delta) \subset U$.

If $s \in V$ then $(s - \delta, s + \delta) \subset V \Rightarrow (s - \delta, s + \delta) \cap U = \emptyset$. □

Corollary 3.6 (The Intermediate Value Theorem). *Let $a < b$ and $f : [a, b] \mapsto \mathbb{R}$ be continuous. If $f(a) < y < f(b)$ then $\exists\ x \in [a, b]$ such that $f(x) = y$.*

Proof. If not, $f^{-1}((-\infty, y))$ and $f^{-1}((y, \infty))$ disconnect $[a, b]$. □

3.3 Path connectedness

Definition 3.7. *Let X be a topological space and let $x, y \in X$. A (continuous) path from x to y is a continuous function $\phi : [a, b] \mapsto X$ such that $\phi(a) = x$ and $\phi(b) = y$.*

Definition 3.8. *X is path connected if $\forall\ x, y \in X, \exists$ a path from x to y.*

Proposition 3.9. *Path-connectedness implies connectedness.*

Proof. Suppose X is a topological space and U and V disconnect X. Let $u \in U$ and $v \in V$. If X is path connected then $\exists$ continuous $\phi : [a, b] \mapsto X$ with $\phi(a) = u$ and $\phi(b) = v$. Then $\phi^{-1}(U)$ and $\phi^{-1}(V)$ disconnect $[a, b]$. □

Definition 3.10. *Let X be a topological space and $\phi : [a, b] \mapsto X$ and $\psi : [c, d] \mapsto X$ be continuous paths with $\phi(b) = \psi(c)$. Then the join of ϕ and ψ, written $\phi \vee \psi$ is defined as $\phi \vee \psi : [a, b + d - c] \mapsto X$*

$$\phi \vee \psi : t \mapsto \begin{cases} \phi(t) & a \leq t < b \\ \psi(t + c - b) & b \leq t \leq b + d - c \end{cases}$$

Definition 3.11. *The reverse of ϕ, written $-\phi$ is the path $-\phi : [-b, -a] \mapsto X$ with $-\phi : t \mapsto \phi(-t)$.*

Let us write $x \to y$ if $\exists$ a continuous path in X from x to y. This is an equivalence relation. Writing $x \xrightarrow{\phi} y$ for "ϕ is a path from x to y", we have that :-

1. $x \xrightarrow{\phi} y \Rightarrow y \xrightarrow{-\phi} x$

2. $x \xrightarrow{\phi} y, y \xrightarrow{\psi} z \Rightarrow x \xrightarrow{\phi\vee\psi} z$

3. the path $\chi\colon [0,1] \mapsto X, \chi(t) = x$ satisfies $x \xrightarrow{\chi} x$.

Thus $\to$ is an equivalence relation. The equivalence classes of $\to$ are called path-components.

Definition 3.12. *Let $X \subset \mathbb{R}^n$. Then a polygonal path is a path $\phi\colon [a,b] \mapsto X$ which is piecewise linear, that is $\exists\ a = x_0 < x_1 < \cdots < x_n = b$ such that $\phi((1-t)x_{i-1} + tx_i) = (1-t)\phi(x_{i-1}) + t\phi(x_i)$ for $t \in [0,1]$ and $1 \le i \le n$. X is said to be polygonally connected if any two points of X can be joined by polygonal paths.*

Write $x \twoheadrightarrow y$ [1] for "$\exists$ a polygonal path in X from x to y". It is easy to see that $\twoheadrightarrow$ is an equivalence relation.

Theorem 3.13. *Let $X \subset \mathbb{R}^n$ be open. Then the following are equivalent :-*

1. *X is connected,*

2. *X is path-connected,*

3. *X is polygonally connected.*

Proof. $3 \Rightarrow 2$ is trivial and $2 \Rightarrow 1$ has been done before. Thus it suffices to show that $1 \Rightarrow 3$.

Suppose X is connected and for $x \in X$ let $U = \{y \in X : x \twoheadrightarrow y\}$. It remains to show that U is both open and closed in X, thus since X is connected, $U = X$.

Now, let $y \in U$. Since X is open $\exists\ \delta > 0$ such that $B_\delta(y) \subset X$. Then for $z \in B_\delta(y)$, $y \twoheadrightarrow z$, thus $x \twoheadrightarrow z$ by transitivity and U is open.

Let $y \in X \setminus U$. Since X is open, $\exists\ \delta > 0$ such that $B_\delta(y) \subset X$. As before, $z \in B_\delta(y) \Rightarrow y \twoheadrightarrow z$. If $x \twoheadrightarrow z$ then $x \twoheadrightarrow y$. Thus $X \setminus U$ is open and U is closed. □

N.B. *The above argument also shows that if $X \subset \mathbb{R}^n$ is open then all path components of X are open.*

Lemma 3.14. *Let $n \ge 2$ and $X \subset \mathbb{R}^n$ be a compact topological space. Then X^c has only open path-components and precisely one of these is unbounded.*

Proof. X^c is an open subset of $\mathbb{R}^n$, so its path components are open. X is bounded so $\exists\ M$ such that $X \subset \{y : \|y\| \le M\}$. It is easy to check that $\{y : \|y\| > M\}$ is path-connected, so it is contained in an unbounded path-component of X^c. The other path-components lie in $\{y : \|y\| \le M\}$ and are therefore bounded. □

[1] Not 100% standard notation.

Chapter 4

Preliminaries to complex analysis

4.1 Paths

Definition 4.1. *A path $\phi\colon [a,b] \mapsto \mathbb{C}$ is smooth if the function ϕ is continuously differentiable, with the appropriate one-sided limits at the endpoints a and b. In particular, ϕ and ϕ' are bounded.*

Two paths $\phi\colon [a,b] \mapsto \mathbb{C}$ and $\psi\colon [c,d] \mapsto \mathbb{C}$ are equivalent if $\exists$ a continuously differentiable function[1] $\gamma\colon [a,b] \mapsto [c,d]$ with $\gamma'(t) > 0\ \forall\ t \in [a,b]$, $\gamma(a) = c$, $\gamma(b) = d$ and $\phi(t) = \psi(\gamma(t))\ \forall\, t \in [a,b]$. Note $\exists\, \delta > 0$ such that $\phi'(t) \geq \delta\ \forall\, t \in [a,b]$. This is easily shown to be an equivalence relation.

Definition 4.2. *If ϕ is a path, $\phi\colon [a,b] \mapsto \mathbb{C}$, the track ϕ^* of ϕ is defined by $\phi^* = \{\phi(t) \in \mathbb{C} : a \leq t \leq b\}$.*

N.B. *Equivalent paths have the same track.*

Definition 4.3. *A path $\phi\colon [a,b] \mapsto \mathbb{C}$ is piecewise smooth if $\exists\ a = x_0 < x_1 < \cdots < x_n = b$ such that the restriction of ϕ to $[x_{i-1}, x_i]$, $\phi\big|_{[x_{i-1},x_i]}$ is smooth. Equivalently, ϕ is piecewise smooth if it can be written as the join of finitely many smooth paths.*

Two piecewise smooth paths $\phi\colon [a,b] \mapsto \mathbb{C}$ and $\psi\colon [c,d] \mapsto \mathbb{C}$ are equivalent if we can find $a = x_0 < x_1 < \cdots < x_n = b$ and $c = y_0 < y_1 < \cdots < y_n = d$ such that for all $1 \leq i \leq n$, $\phi\big|_{[x_{i-1},x_i]}$ and $\psi\big|_{[y_{i-1},y_i]}$ are equivalent.

Definition 4.4. *A piecewise smooth path $\phi\colon [a,b] \mapsto \mathbb{C}$ is closed if $\phi(a) = \phi(b)$. It is simple if $\phi(x) = \phi(y) \Rightarrow \{x,y\} \subset \{a,b\}$ or $x = y$.*

4.2 Complex Integration

A function $f\colon [a,b] \mapsto \mathbb{C}$ is said to be Riemann integrable if both its real and imaginary parts are Riemann integrable. The integral is defined to be

$$\int_a^b f(t)\mathrm{d}t = \int_a^b \Re(f(t))\mathrm{d}t + \imath \int_a^b \Im(f(t))\mathrm{d}t.$$

[1] Including appropriate one-sided limits at the endpoints.

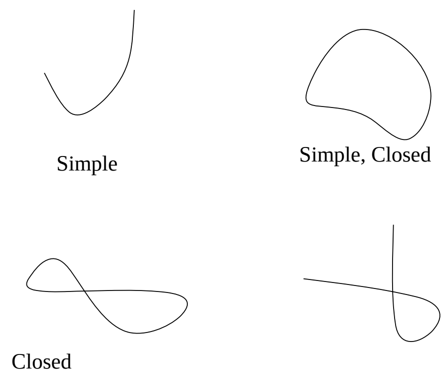


Figure 4.1: Examples of paths

Lemma 4.5. *Let $f\colon [a,b] \mapsto \mathbb{C}$ be continuous. Then*

$$\left|\int_a^b f(t)\mathrm{d}t\right| \leq \int_a^b |f(t)|\,\mathrm{d}t.$$

Proof. For suitable $\theta \in \mathbb{R}$,

$$\begin{aligned}\left|\int_a^b f(t)\mathrm{d}t\right| &= e^{\imath\theta}\int_a^b f(t)\mathrm{d}t = \int_a^b e^{\imath\theta} f(t)\mathrm{d}t \\ &= \Re\int_a^b e^{\imath\theta} f(t)\mathrm{d}t = \int_a^b \Re(e^{\imath\theta} f(t))\mathrm{d}t \\ &\leq \int_a^b |f(t)|\,\mathrm{d}t \quad \text{using the real result.}\end{aligned}$$

□

4.3 Domains

Definition 4.6. *A domain is a connected open subset of $\mathbb{C}$.*

Examples. *Some examples of domains are*

1. $\mathbb{C}$,
2. $\Delta = \{z : |z| < 1\}$,
3. $\mathbb{C} \setminus \{0\}$,
4. $\{z : a < |z| < b\}$.

Definition 4.7. *Given $x, y \in \mathbb{C}$, let $[x \to y]$ be the path $\phi\colon [0,1] \mapsto \mathbb{C}$, $\phi(t) = (1-t)x + ty$.*

Definition 4.8. *A domain D is convex if $x, y \in D \Rightarrow [x \to y]^* \subset D$.*

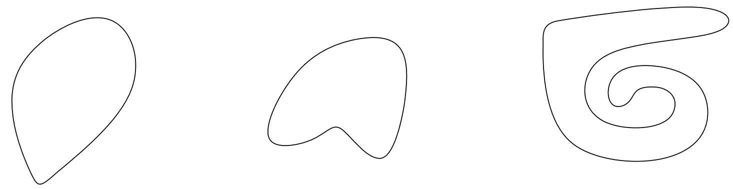


Figure 4.2: Examples of domains

Definition 4.9. *A domain D is star-shaped if $\exists\, z_0 \in D$ such that $[z_0 \to z]^* \subset D \;\forall\, z \in D$.*

A star-shaped domain is sometimes called a star-domain. Note that every (non-empty) convex domain is star-shaped.

4.4 Path Integrals

Definition 4.10. *Let D be a domain, $f\colon D \mapsto \mathbb{C}$ be continuous and $\phi\colon [a,b] \mapsto D$ be a smooth path. Then the integral of f along ϕ is defined as*

$$\int_\phi f(z)\mathrm{d}z = \int_a^b f(\phi(t))\phi'(t)\mathrm{d}t.$$

If ϕ is piecewise smooth, then pick $a = x_0 < \cdots < x_n = b$ such that $\phi\big|_{[x_{i-1},x_i]}$ is smooth and then

$$\int_\phi f(z)\mathrm{d}z = \sum_{i=1}^n \int_{x_{i-1}}^{x_i} f(\phi(t))\phi'(t)\mathrm{d}t.$$

Lemma 4.11. *Equivalent paths give the same integral.*

Proof. Let D be a domain and $\phi\colon [a,b] \mapsto D$ and $\psi\colon [c,d] \mapsto D$ be equivalent smooth paths. Let $f\colon D \mapsto \mathbb{C}$ be continuous and $\gamma\colon [a,b] \mapsto [c,d]$ give the equivalence. Then

$$\begin{aligned}\int_\psi f(z)\mathrm{d}z &= \int_c^d f(\psi(s))\psi'(s)\mathrm{d}s = \int_a^b f(\psi(\gamma(t)))\psi'(\gamma(t))\gamma'(t)\mathrm{d}t \\ &= \int_a^b f(\phi(t))\phi'(t)\mathrm{d}t = \int_\phi f(z)\mathrm{d}z.\end{aligned}$$

If ϕ and ψ are piecewise smooth write $\phi = \phi_1 \vee \cdots \vee \phi_n$ and $\psi = \psi_1 \vee \cdots \vee \psi_n$, with ϕ_i and ψ_i smooth, ϕ_i equivalent to ψ_i. Then

$$\int_\phi f(z)\mathrm{d}z = \sum_{i=1}^n \int_{\phi_i} f(z)\mathrm{d}z = \sum_{i=1}^n \int_{\psi_i} f(z)\mathrm{d}z = \int_\psi f(z)\mathrm{d}z.$$

□

Example. *Let D be $\mathbb{C} \setminus \{0\}$ and $f(z) = z^n$ for some $n \in \mathbb{Z}$, $\phi\colon [0, 2\pi] \mapsto D$ be $t \mapsto e^{\imath t}$. Then*

$$\int_\phi f(z)\mathrm{d}z = \begin{cases} 2\pi\imath & \text{if } n = -1 \\ 0 & \text{otherwise.} \end{cases}$$

Proof.

$$\begin{aligned} \int_\phi f(z)\mathrm{d}z &= \int_0^{2\pi} e^{n\imath t}\imath e^{\imath t}\mathrm{d}t = \imath \int_0^{2\pi} e^{(n+1)\imath t}\mathrm{d}t \\ &= \begin{cases} \left[\frac{e^{(n+1)\imath t}}{n+1}\right]_0^{2\pi} = 0 & \text{if } n \neq -1 \\ 2\pi\imath & \text{if } n = -1. \end{cases} \end{aligned}$$

□

Definition 4.12. *Let D be a domain and $\phi\colon [a, b] \mapsto D$ be a smooth path. Then the length of ϕ, $L(\phi)$ is defined as*

$$L(\phi) = \int_a^b |\phi'(t)|\, \mathrm{d}t.$$

Lemma 4.13. *Let D be a domain, $f\colon D \mapsto \mathbb{C}$ be continuous and $\phi\colon [a, b] \mapsto D$ be a smooth path. Then*

$$\left|\int_\phi f(z)\mathrm{d}z\right| \leq \sup_{z\in\phi^*} |f(z)|\, L(\phi).$$

Proof.

$$\begin{aligned} \left|\int_\phi f(z)\mathrm{d}z\right| &= \left|\int_a^b f(\phi(t))\phi'(t)\mathrm{d}t\right| \\ &\leq \int_a^b |f(\phi(t))|\, |\phi'(t)|\, \mathrm{d}t \\ &\leq \sup_{z\in\phi^*} |f(z)|\, L(\phi) \quad \text{by real result.} \end{aligned}$$

□

Remark. *The above generalises easily to piecewise smooth paths.*

Henceforth, all paths are piecewise smooth unless otherwise stated.

Proposition 4.14 (Fundamental Theorem of Calculus). *Let D be a domain and let $f\colon D \mapsto \mathbb{C}$ be continuous. Suppose f has an antiderivative F (i.e. a function $F(z)$ such that $F'(z) = f(z)\ \forall\ z \in D$). Let $\phi\colon [a, b] \mapsto D$ be a path. Then*

$$\int_\phi f(z)\mathrm{d}z = F(\phi(b)) - F(\phi(a)).$$

Proof. If ϕ is smooth, then

$$\int_\phi f(z)\mathrm{d}z = \int_a^b f(\phi(t))\phi'(t)\mathrm{d}t = \int_a^b (F\circ\phi)'(t)\mathrm{d}t = F\circ\phi(b) - F\circ\phi(a).$$

In general if $a = x_0 < x_1 < \cdots < x_n = b$ and $\phi\big|_{[x_{i-1},x_i]}$ is smooth, then the above argument gives that

$$\int_\phi f(z)\mathrm{d}z = \sum_{i=1}^n (F(\phi(x_i)) - F(\phi(x_{i-1}))) = F(\phi(b)) - F(\phi(a)).$$

□

Corollary 4.15. *If D is a domain, $f\colon D \mapsto \mathbb{C}$ is continuous with antiderivative F and ϕ is a closed path, then*

$$\int_\phi f(z)\mathrm{d}z = 0.$$

Proof. Immediate. □

Lemma 4.16. *Let D be a star-domain and $f\colon D \mapsto \mathbb{C}$ be continuous. Then the following are equivalent.*

1. *f has an antiderivative F on D.*
2. *$\int_\phi f(z)\mathrm{d}z = 0$ for all closed paths ϕ in D.*
3. *$\int_{\partial T} f(z)\mathrm{d}z = 0$ for the boundary ∂T of any triangle T such that $T \subset D$* [2].

Proof. It is enough to do $3 \Rightarrow 1$. Take $z_0 \in D$ such that $[z_0 \to z]^* \subset D\ \forall\, z \in D$ and define

$$F(z) = \int_{[z_0\to z]^*} f(w)\mathrm{d}w.$$

Then take T to be the triangle with vertices z_0, z and $z+h$. Since D is open, $[z \to z+h]^* \subset D$ for $|h|$ sufficiently small which gives that $T \subset D$. Now

$$F(z+h) - F(z) = \int_{[z\to z+h]^*} f(w)\mathrm{d}w, \text{ so that}$$

$$\begin{aligned}|F(z+h) - F(z) - hf(z)| &= \left|\int_{[z\to z+h]^*} (f(w) - f(z))\mathrm{d}w\right| \\ &\le \sup_{w\in[z\to z+h]} |f(w) - f(z)|\,|h|\,.\end{aligned}$$

Choose $\delta > 0$ such that $|h| < \delta \Rightarrow |f(z+h) - f(z)| < \epsilon$. Then

$$|F(z+h) - F(z) - hf(z)| \le \epsilon\,|h|\,.$$

□

[2] Including the boundary and interior.

Chapter 5

Cauchy's theorem and its consequences

5.1 Cauchy's theorem

Definition 5.1. *Let D be a domain and $f\colon D \mapsto \mathbb{C}$ be continuous. f is analytic (or holomorphic)*[1] *if f is differentiable at z $\forall z \in D$.*

Theorem 5.2 (Cauchy's theorem for triangles). *Let D be a domain and T be a triangle lying entirely in D. If $f\colon D \mapsto \mathbb{C}$ is analytic, then*

$$\int_{\partial T} f(z)\mathrm{d}z = 0.$$

Proof. Let $\eta = \left|\int_{\partial T} f(z)\mathrm{d}z\right|$ and let $l = L(\partial T)$. Now let $T_0 = T$. We can split T into 4 equally sized triangles T^1, T^2, T^3, T^4 as shown, with all boundaries oriented in the same direction as that of T.

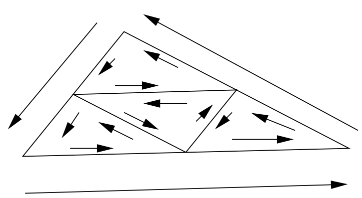


Figure 5.1: Splitting up the triangle

Since the contributions from internal edges cancel,

$$\int_{\partial T} f(z)\mathrm{d}z = \sum_{i=1}^{4} \int_{\partial T^i} f(z)\mathrm{d}z,$$

and $\exists\, i \leq 4$ such that

$$\left|\int_{\partial T^i} f(z)\mathrm{d}z\right| \geq \frac{\eta}{4}.$$

[1]Outside Cambridge, an analytic function is one which has a power series expansion and a holomorphic function is $\mathbb{C}$ differentiable on a domain.

Put $T_1 = T^i$ for this i and repeat the process. We produce a sequence $T_0, T_1, T_2, \ldots$ such that

$$\left| \int_{\partial T_n} f(z) \mathrm{d}z \right| \geq \frac{\eta}{4^n} \text{ and}$$

$$L(\partial T_n) = \frac{l}{2^n}.$$

Since the T_n are closed, we can find $z_0 \in \bigcap_{i=1}^{\infty} T_n$. As f is differentiable at z_0, $\forall \, \epsilon > 0, \exists \, \delta > 0$ such that

$$|w - z_0| < \delta \Rightarrow |f(w) - f(z_0) - (w - z_0) f'(z_0)| < \epsilon \, |w - z_0| \, .$$

Pick n such that $T_n \subset B_\delta(z_0)$. Then

$$\begin{aligned} \left| \int_{\partial T_n} f(z) \mathrm{d}z \right| &= \left| \int_{\partial T_n} (f(z) - f(z_0) - (z - z_0) f'(z_0)) \mathrm{d}z \right| \\ &\leq L(\partial T_n) \epsilon \sup_{z \in \partial T_n} |z - z_0| \\ &\leq \epsilon L(\partial T_n)^2. \end{aligned}$$

But $\left| \int_{\partial T_n} f(w) \mathrm{d}w \right| \geq 4^{-n} \eta l^2$. This gives that $\eta < \epsilon$. But $\epsilon > 0$ is arbitrary, so $\eta = 0$. □

Corollary 5.3 (Cauchy's Theorem for a star-domain). *Let D be a star-domain and $f \colon D \mapsto \mathbb{C}$ be analytic. Then*

$$\int_\phi f(z) \mathrm{d}z = 0 \textit{ for all closed paths } \phi \textit{ in } D.$$

Proof. Result true for triangles. Thus f has an anti-derivative and thus

$$\int_\phi f(z) \mathrm{d}z = 0.$$

□

5.2 Homotopy

Definition 5.4. *Let $\phi \colon [0, 1] \mapsto D$ and $\psi \colon [0, 1] \mapsto D$ be piecewise smooth closed paths in a domain D. A homotopy from ϕ to ψ is a function $\gamma \colon [0, 1]^2 \mapsto D$ such that*

1. *γ is continuous,*
2. *$\gamma(0, t) = \phi(t) \, \forall \, t \in [0, 1]$,*
3. *$\gamma(1, t) = \psi(t) \, \forall \, t \in [0, 1]$,*
4. *$\forall \, s \in [0, 1]$, the path $\gamma_s(t)$ defined by $\gamma_s(t) = \gamma(s, t)$ is closed and piecewise smooth.*

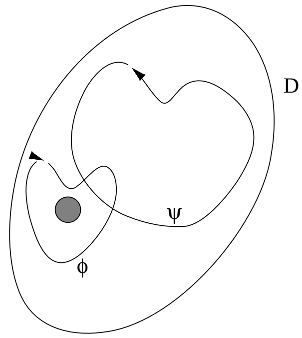


Figure 5.2: Two non-homotopic paths.

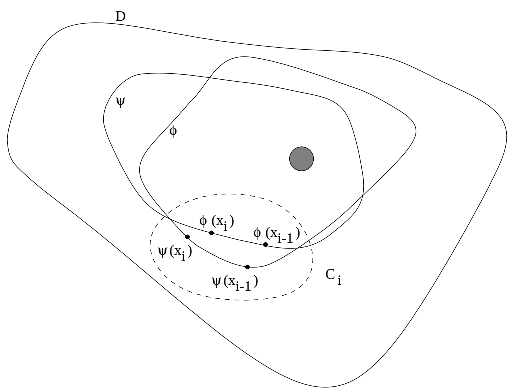


Figure 5.3: Elementary deformation

Definition 5.5. *ψ is said to be an elementary deformation of ϕ if $\exists\ 0 = x_0 < x_1 < \cdots < x_n = 1$ and convex open subsets $C_1, \ldots, C_n \subset D$ such that $x_{i-1} \leq t \leq x_i \Rightarrow \phi(t) \in C_i, \psi(t) \in C_i$.*

Lemma 5.6. *Let D be a domain, $f\colon D \mapsto \mathbb{C}$ be analytic, $\phi\colon [0,1] \mapsto D$ be a closed path and ψ be an elementary deformation of ϕ. Then*

$$\int_\phi f(z)\mathrm{d}z = \int_\psi f(z)\mathrm{d}z.$$

Proof. Let ϕ_i and ψ_i be the restrictions to $[x_{i-1}, x_i]$ of ϕ and ψ respectively. Let $\gamma_i = [\phi(x_i) \to \psi(x_i)]$. By Cauchy's theorem for a star-domain,

$$\int_{\phi_i} f(z)\mathrm{d}z + \int_{\gamma_i} f(z)\mathrm{d}z - \int_{\psi_i} f(z)\mathrm{d}z - \int_{\gamma_{i-1}} f(z)\mathrm{d}z = 0.$$

Now summing from $i = 1 \ldots n$ gives that

$$\int_\phi f(z)\mathrm{d}z = \int_\psi f(z)\mathrm{d}z.$$

□

Proposition 5.7. *Let D be a domain and let $\phi\colon [0,1] \mapsto D$ and $\psi\colon [0,1] \mapsto D$ be homotopic. Then $\exists\, \phi = \phi_0, \phi_1, \dots, \phi_n = \psi$ such that ϕ_i is an elementary deformation of ϕ_{i-1}.*

Proof. Let $\gamma\colon [0,1]^2 \mapsto D$ be a homotopy from ϕ to ψ. $[0,1]^2$ is compact, so $\gamma([0,1]^2)$ is a compact subset of D. Now $\mathbb{C} \setminus D$ is closed and disjoint from $\gamma([0,1]^2)$ so $\exists\, \epsilon > 0$ such that $\forall\, z \in \gamma([0,1]^2)$ and $w \notin D, |z-w| \geq \epsilon$. Thus $\forall (s,t) \in [0,1]^2$, $B_\epsilon(\gamma(s,t)) \subset D$. Also, γ is uniformly continuous on $[0,1]^2$, so $\exists\, \delta > 0$ such that

$$\left((s-s')^2 + (t-t')^2\right)^{1/2} < \delta \Rightarrow |\gamma(s,t) - \gamma(s',t')| < \epsilon.$$

Now pick $n \in \mathbb{N}$ such that $\frac{2}{n} < \delta$ and let $\phi_i = \gamma_{\frac{i}{n}}, i = 0, \dots, n$. Let $x_j = \frac{j}{n}$ and $C_{ij} = B_\epsilon(\gamma(x_i, x_j))$.

But if $\frac{i-1}{n} \leq s \leq \frac{i}{n}$ and $\frac{j-1}{n} \leq t \leq \frac{j}{n}$ then

$$\left((s-s')^2 + (t-t')^2\right)^{1/2} < \frac{2}{n} < \delta \Rightarrow |\gamma(s,t) - \gamma(i/n, j/n)| < \epsilon \Rightarrow \gamma(s,t) \in C_{ij}.$$

Thus ϕ_i is an elementary deformation of ϕ_{i-1}. □

Corollary 5.8. *Let D be a domain, $f\colon D \mapsto \mathbb{C}$ be analytic and ϕ, ψ be homotopic closed paths in D. Then*

$$\int_\phi f(z)\mathrm{d}z = \int_\psi f(z)\mathrm{d}z.$$

Proof. Immediate from above. □

Definition 5.9. *Let D be a domain. A closed path ϕ is contractible if it is homotopic to a constant path.*

Definition 5.10. *A domain D is simply connected if every closed path is contractible.*

Corollary 5.11 (Cauchy's theorem for a simply connected domain). *Let D be a domain and $f\colon D \mapsto \mathbb{C}$ be analytic. If the closed path ϕ is contractible, then*

$$\int_\phi f(z)\mathrm{d}z = 0.$$

If D is simply connected then

$$\int_\phi f(z)\mathrm{d}z = 0 \text{ for all closed paths } \phi.$$

Proof. Immediate. □

Notation.

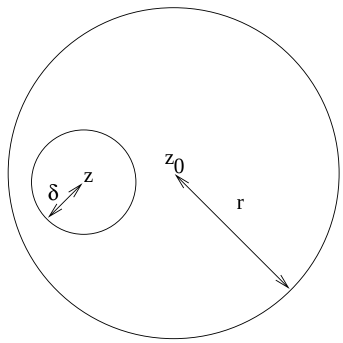
$$B_r(z_0) \equiv B(z_0, r) \equiv \{z \in \mathbb{C} : |z - z_0| < r\}$$
$$\text{thus } \overline{B_r(z_0)} \equiv \overline{B(z_0,r)} \equiv \{z \in \mathbb{C} : |z - z_0| \leq r\}.$$

$C(z_0, r) \equiv C_r(z_0)$ *is the path* $t \mapsto z_0 + re^{2\pi\imath t}$ *for* $t \in [0,1]$.

5.3 Consequences of Cauchy's Theorem

Theorem 5.12 (Cauchy's Integral Formula). *Let D be a domain and let $f\colon D \mapsto \mathbb{C}$ be analytic. Let z_0, r be such that $\overline{B_r(z_0)} \subset D$. Then $\forall\, z \in B_r(z_0)$,*

$$f(z) = \frac{1}{2\pi\imath}\int_{C_r(z_0)} \frac{f(w)}{w-z}\mathrm{d}w.$$



Proof. Take $\epsilon > 0$. Let $\delta > 0$ be such that $\overline{B_\delta(z)} \subset B_r(z_0)$ and $|w-z| = \delta \Rightarrow |f(w) - f(z)| \leq \epsilon$. Then

$$\left| f(z) - \frac{1}{2\pi\imath}\int_{C_r(z_0)} \frac{f(w)}{w-z}\mathrm{d}w \right| = \left| f(z) - \frac{1}{2\pi\imath}\int_{C_\delta(z)} \frac{f(w)}{w-z}\mathrm{d}w \right|$$

since $\gamma(s,t) = (1-s)(z_0 + re^{2\pi\imath t}) + s(z + \delta e^{2\pi\imath t})$ is a homotopy from $C_r(z_0)$ to $C_\delta(z)$ in $D \setminus \{z\}$ in which $\frac{f(w)}{w-z}$ is analytic. Thus

$$\begin{aligned} &= \left| \frac{1}{2\pi\imath}\int_{C_\delta(z)} \frac{f(z)-f(w)}{w-z}\mathrm{d}w \right| \\ &\leq \frac{1}{2\pi}\frac{2\pi\delta\epsilon}{\delta} = \epsilon. \end{aligned}$$

But $\epsilon > 0$ is arbitrary, so result follows. □

Remark. *Note that the proof of Cauchy's integral formula given does not need the full strength of homotopy invariance, since $C_\delta(z)$ is clearly an elementary deformation of $C_r(z_0)$*

Theorem 5.13 (Liouville's Theorem). *Every bounded entire function is constant.*

Proof. Let $f\colon \mathbb{C} \mapsto \mathbb{C}$ be analytic and $|f(z)| \leq M \;\forall\, z \in \mathbb{C}$. Take $z_1, z_2 \in \mathbb{C}$ and let $R \geq 2\max\{|z_1|, |z_2|\}$. Then

$$\begin{aligned}
|f(z_1) - f(z_2)| &= \left| \frac{1}{2\pi\imath} \int_{C_R(0)} \left(\frac{f(w)}{w - z_1} - \frac{f(w)}{w - z_2} \right) \mathrm{d}w \right| \\
&= \left| \frac{1}{2\pi\imath} \int_{C_R(0)} \frac{f(w)(z_1 - z_2)}{(w - z_1)(w - z_2)} \mathrm{d}w \right| \\
&\le \frac{1}{2\pi} \frac{2\pi R M \left| z_1 - z_2 \right|}{\left(\frac{R}{2}\right)^2} \\
&= \frac{4M \left| z_1 - z_2 \right|}{R}.
\end{aligned}$$

But R can be arbitrarily large, so result follows. □

Theorem 5.14 (The Fundamental Theorem of Algebra). *Every non-constant polynomial has at least one root in* $\mathbb{C}$*.*

Proof. Let p be a non-constant polynomial and suppose that p has no roots. Then the function $\frac{1}{p(z)}$ is analytic on $\mathbb{C}$. Suppose that

$$p(z) = a_n z^n + \cdots + a_0, \text{ with } a_n \neq 0.$$

Then if

$$|z| \ge \max \left\{ 1, 2 \frac{|a_{n-1}| + \cdots + |a_0|}{|a_n|} \right\},$$

$$\begin{aligned}
|p(z)| &\ge |a_n| \, |z|^n - (|a_{n-1}| + \cdots + |a_0|) \, |z|^{n-1} \\
&\ge \frac{1}{2} |a_n| \, |z|^n \ge \frac{1}{2} |a_n| \, .
\end{aligned}$$

So $\exists\ M$ such that $|z| \ge M \Rightarrow \left| \frac{1}{p(z)} \right| \le \frac{2}{a_n}$. Now $\frac{1}{p(z)}$ is continuous and $\overline{B_M(0)}$ is compact, so $\frac{1}{p}$ is bounded in $\overline{B_M(0)}$. Hence $\frac{1}{p}$ bounded on all of $\mathbb{C}$ and thus constant. This is a contradiction. □

Proposition 5.15. *Let g be a continuous function from $\{z \in \mathbb{C} : |z - z_0| = r\}$ to $\mathbb{C}$. Then*

$$f(z) = \int_{C_r(z_0)} \frac{g(w)}{(w - z)^n} \mathrm{d}w$$

is analytic on $B_r(z_0)$ and

$$f'(z) = n \int_{C_r(z_0)} \frac{g(w)}{(w - z)^{n+1}} \mathrm{d}w$$

Proof. Let $2\epsilon = r - |z - z_0|$ such that $|w - z_0| = r \Rightarrow |w - z| \ge 2\epsilon$. Now

$$\left|\frac{1}{(w-z-h)^n}-\frac{1}{(w-z)^n}\right| =$$

$$\begin{aligned}
&\left|\left(\frac{1}{w-z-h}-\frac{1}{w-z}\right)\sum_{k=0}^{n-1}\frac{1}{(w-z-h)^k(w-z)^{n-1-k}}\right| \\
&=\left|\frac{h}{(w-z-h)(w-z)}\sum_{k=0}^{n-1}\frac{1}{(w-z-h)^k(w-z)^{n-1-k}}\right| \\
&\leq \frac{hn}{\epsilon^{n+1}} \\
&\to 0 \text{ as } |h|\to 0 \text{ independently of } w.
\end{aligned}$$

When $|h| \leq \epsilon$,

$$\left|f(z+h)-f(z)-hn\int_{C_r(z_0)}\frac{g(w)}{(w-z)^{n+1}}\mathrm{d}w\right| =$$

$$\left|\int_{C_r(z_0)} hg(w)\left(\frac{1}{(w-z-h)(w-z)}\sum_{k=0}^{n-1}\frac{1}{(w-z-h)^k(w-z)^{n-1-k}}\right)\mathrm{d}w\right|$$

Using the same estimate as above, the bit in brackets converges to 0 as $h \to 0$. Since $g(w)$ is bounded on $C_r(z_0)^*$ for $|h|$ sufficiently small the whole integral is at most $\epsilon\,|h|$. □

Corollary 5.16. *Let D be a domain and $f\colon D \mapsto \mathbb{C}$ be analytic. Then f is infinitely differentiable inside $B_r(z_0)$, $\overline{B_r(z_0)} \subset D$ and $\forall\, z \in B_r(z_0)$,*

$$f^{(n)}(z) = \frac{n!}{2\pi\imath}\int_{C_r(z_0)}\frac{f(w)}{(w-z)^{n+1}}\mathrm{d}w.$$

Proof. The case $n = 0$ is Cauchy's integral formula. If we have it for n, then the proposition gives it for $n+1$. □

Theorem 5.17 (Morera's Theorem). *Let D be a star-shaped domain and $f\colon D \mapsto \mathbb{C}$ be continuous. If*

$$\int_{\partial T} f(z)\mathrm{d}z = 0$$

for all triangles $T \subset D$ then f is analytic.

Proof. The condition implies that f has an antiderivative F, which is analytic. F is therefore infinitely differentiable, so f is analytic. □

Remark. *Now let D be an arbitrary domain and let $z \in D$. Since $\exists\,\epsilon > 0$ such that $B_\epsilon(z) \subset D$ and $B_\epsilon(z)$ is star-shaped, one can easily extend Morera's Theorem to any domain.*

Chapter 6

Power Series

6.1 Analyticity and Holomorphy

Lemma 6.1. *Consider a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$. If this sum converges for some z with $|z-z_0| = \rho$, then it converges for all w with $|w-z_0| < \rho$ and for any $r < \rho$, the convergence is uniform in $\overline{B_r(z_0)}$.*

Proof. Since $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges, then $\exists\ M$ such that $|a_n|\,|z-z_0|^n = |a_n|\,\rho^n \leq M\ \forall\, n$. If $|w-z_0| < \rho$, then

$$\begin{aligned}\left|\sum_{n=N}^{\infty} a_n(w-z_0)^n\right| &\leq M \sum_{n=N}^{\infty}\left|\frac{w-z_0}{\rho}\right|^n \\ &\leq \sum_{n=N}^{\infty}\left(\frac{r}{\rho}\right)^n \\ &= M\left(\frac{r}{\rho}\right)^n \frac{1}{1-\frac{r}{\rho}} \\ &\to 0 \text{ independently of } w.\end{aligned}$$

□

Definition 6.2. *The radius of convergence of a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is*

$$R = \sup\{r : \exists\ z \textit{ such that } |z-z_0| < r \textit{ and } \sum_{n=0}^{\infty} a_n(z-z_0)^n \textit{ converges.}\}.$$

Lemma 6.3. *Let D be a domain, $\phi\colon [a,b] \mapsto D$ be a path and $f_n\colon D \mapsto \mathbb{C}$ be continuous. Suppose $f_n \to f$ uniformly on ϕ^*. Then*

$$\int_\phi f_n(z)\mathrm{d}z \to \int_\phi f(z)\mathrm{d}z.$$

Proof. Let $\epsilon > 0$. Then $\exists\ N$ such that $\forall\ n \geq N\ \forall\ z \in \phi^*$, $|f_n(z) - f(z)| \leq \frac{\epsilon}{L(\phi)}$. Then

$$\begin{aligned}\left|\int_\phi f_n(z)\mathrm{d}z - \int_\phi f(z)\mathrm{d}z\right| &= \left|\int_\phi f_n - f\mathrm{d}z\right| \\ &\leq \epsilon.\end{aligned}$$

But $\epsilon > 0$ is arbitrary, so result follows. □

Lemma 6.4. *Let* $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ *and* $g(z) = \sum_{n=0}^{\infty} b_n(z-z_0)^n$*. Suppose there exists a sequence* $(z_k)_{k=1}^{\infty} \to z_0$*,* $z_k \neq z_0$ *and* $f(z_k) = g(z_k)$*. Then* $a_n = b_n$ *for all* n*.*

Proof. Suppose otherwise. Then let N be minimal such that $a_N \neq b_N$. Now let $c_n = a_n - b_n$. Then

$$\begin{aligned} f(z) - g(z) &= \sum_{n=N}^{\infty} c_n(z-z_0)^n \\ &= (z-z_0)^n \left(c_N + (z-z_0) \sum_{n=N+1}^{\infty} c_n (z-z_0)^{n-N-1} \right). \end{aligned}$$

For $|z_0 - z_k|$ sufficiently small, then

$$|z_k - z_0| \left| \sum_{n=N+1}^{\infty} c_n (z-z_0)^{n-N-1} \right| < \frac{1}{2} |c_N|$$

and thus $f(z_k) - g(z_k) \neq 0$. □

Lemma 6.5. *Let* $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ *with radius of convergence* R*. Then* f *is analytic in* $B_R(z_0)$ *and* $f'(z) = \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1}$*.*

Proof. Let $z \in B_R(z_0)$. Pick r such that $|z-z_0| < r < R$. Let

$$f_N(z) = \sum_{n=0}^{N} a_n (z-z_0)^n.$$

Then $f_N \to f$ uniformly on $\overline{B_r(z_0)}$. Since $|w-z_0| = r \Rightarrow |w-z| \geq r - |z-z_0|$, we have

$$\frac{f_N(w)}{w-z} \to \frac{f(w)}{w-z} \quad \text{and} \quad \frac{f_N(w)}{(w-z)^2} \to \frac{f(w)}{(w-z)^2} \text{ uniformly for } |w-z_0| = r.$$

But

$$f_N(z) = \frac{1}{2\pi\imath} \int_{C_r(z_0)} \frac{f_N(w)}{w-z} \mathrm{d}w \to \frac{1}{2\pi\imath} \int_{C_r(z_0)} \frac{f(w)}{w-z} \mathrm{d}w.$$

Therefore

$$f(z) = \frac{1}{2\pi\imath} \int_{C_r(z_0)} \frac{f(w)}{w-z} \mathrm{d}w.$$

Hence f is differentiable at z and

$$f'(z) = \frac{1}{2\pi\imath} \int_{C_r(z_0)} \frac{f(w)}{(w-z)^2} \mathrm{d}w.$$

Also,

$$f_N'(z) = \frac{1}{2\pi\imath} \int_{C_r(z_0)} \frac{f_N(w)}{(w-z)^2} \mathrm{d}w \to \frac{1}{2\pi\imath} \int_{C_r(z_0)} \frac{f(w)}{(w-z)^2} \mathrm{d}w = f'(z).$$

Hence $f'(z) = \sum_{n=1}^{\infty} n a_n (z-z_0)^n$ as claimed. □

Theorem 6.6 (Taylor's Theorem). *Let D be a domain and then $f\colon D \mapsto \mathbb{C}$ be analytic. Let $z_0 \in D$ and let R be such that $B_R(z_0) \subset D$. Then there exist unique coefficients $(a_n)_{n=0}^{\infty}$ such that*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \ \forall\, z \in B_R(z_0).$$

Proof. Let $z \in B_R(z_0)$ and let r be such that $|z - z_0| < r < R$. Then by Cauchy's integral formula,

$$\begin{aligned} f(z) &= \frac{1}{2\pi\imath} \int_{C_r(z_0)} \frac{f(w)}{w - z} \mathrm{d}w \\ &= \frac{1}{2\pi\imath} \int_{C_r(z_0)} \frac{f(w)}{(w - z_0) - (z - z_0)} \mathrm{d}w \\ &= \frac{1}{2\pi\imath} \int_{C_r(z_0)} \frac{f(w)}{w - z_0} \frac{1}{1 - \frac{z - z_0}{w - z_0}} \mathrm{d}w \\ &= \frac{1}{2\pi\imath} \int_{C_r(z_0)} \frac{f(w)}{w - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{w - z_0} \right)^n \mathrm{d}w. \end{aligned}$$

As $\left| \frac{z - z_0}{w - z_0} \right| < 1$, the convergence is uniform, so can exchange the sum and integral to get

$$f(z) = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{2\pi\imath} \int_{C_r(z_0)} \frac{f(w)}{(w - z_0)^{n+1}} \mathrm{d}w.$$

To get uniqueness use above lemma. □

Theorem 6.7 (Identity Theorem). *Let D be a domain and $f, g\colon D \mapsto \mathbb{C}$ be analytic. Suppose $z_k \to z_0 (\in D)$, $z_k \neq z_0$ and $f(z_k) = g(z_k)$ for all k. Then $f(z) = g(z)\ \forall\, z \in D$. In particular, setting $g \equiv 0$ gives that the zeros of a non-constant analytic function are isolated.*

Proof. Define $U = \{z \in D : f^{(n)}(z) = g^{(n)}(z)\ \forall n\}$. Now $U \neq \emptyset$ since $z_0 \in U$ as the earlier result on uniqueness of power series implies that the Taylor expansions of f and g at z_0 are the same.

Now U is closed, since

$$U = \bigcap_{n=0}^{\infty} \left(f^{(n)} - g^{(n)} \right)^{-1} (\{0\}).$$

If $z \in U$, then the Taylor expansions of f and g agree at z, so $f \equiv g$ in some $B_\delta(z)$ and then for $y \in B_\delta(z)$, f and g must have the same n^{th} derivatives at y. Thus U is open and since D is connected, $U = D$. □

Proposition 6.8. *Let D be a domain, $z_0 \in D$ and $f\colon D \mapsto \mathbb{C}$ be analytic such that $f \not\equiv 0$. Then there exist a unique $k \geq 0$ and a unique analytic function g such that $g(z_0) \neq 0$ and $f(z) = (z - z_0)^k g(z)$.*

Proof. Let the Taylor expansion of f at z_0 be $\sum_{n=0}^{\infty} a_n(z-z_0)^n$. Now choose N minimal such that $a_N \neq 0$. (This is do-able since $f \not\equiv 0$.) Thus we can write

$$f(z) = (z-z_0)^N \sum_{n=0}^{\infty} a_{n+N}(z-z_0)^n$$

in some $B_\delta(z_0)$. Set

$$g(z) = \begin{cases} \sum_{n=0}^{\infty} a_{n+N}(z-z_0)^n & \text{if } |z-z_0| < \delta \\ (z-z_0)^{-N} f(z) & \text{if } z \neq z_0. \end{cases}$$

These two cases agree when $0 < |z-z_0| < \delta$ and $f(z) = (z-z_0)^N g(z)$, $g(z_0) = a_N \neq 0$. Now if $f(z) = (z-z_0)^{k_1} g_1(z) = (z-z_0)^{k_2} g_2(z)\ \forall\, z \in D$, take $k_1 \leq k_2$ without loss of generality. Then for $z \neq z_0$ we have

$$g_1(z) = (z-z_0)^{k_2-k_1} g_2(z).$$

Thus if $k_1 < k_2$, $g_1 \to 0$ as $z \to z_0$ and so $g(z_0) = 0$. Thus $g_1(z) = g_2(z)$ if $z \neq z_0$ and hence $g_1 \equiv g_2$. □

Theorem 6.9 (Riemann's Removable Singularity Theorem). *Consider a domain D with $z_0 \in D$. Let $f\colon D \setminus \{z_0\} \mapsto \mathbb{C}$ be analytic. Then if f is bounded near z_0 (i.e. $\exists\, \delta > 0, M$ such that $z \in B_\delta(z_0) \Rightarrow |f(z)| \leq M$), $f \to a$ as $z \to z_0$ and the function*

$$g(z) = \begin{cases} f(z) & z \neq z_0 \\ a & z = z_0 \end{cases}$$

is analytic.

Proof. Define $h\colon D \mapsto \mathbb{C}$ by

$$h(z) = \begin{cases} (z-z_0)^2 f(z) & z \neq z_0 \\ 0 & z = z_0. \end{cases}$$

This is differentiable at $z \neq z_0$, and also

$$\left| \frac{h(z)-h(z_0)}{z-z_0} \right| \leq M\,|z-z_0| \text{ when } |z-z_0| < \delta.$$

Hence h is analytic and so has Taylor series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$. Now $h(z_0) = h'(z_0) = 0$, so if we define $g(z) = \sum_{n=0}^{\infty} a_{n+2}(z-z_0)^n$, then $g(z) = f(z)$ for $z \neq z_0$. Hence $g(z) \to a$ as $z \to z_0$, so $f(z) \to a$ as $z \to z_0$. □

Proposition 6.10. *Let D be a domain, $z_0 \in D$ and $f\colon D \setminus \{z_0\} \mapsto \mathbb{C}$ be analytic. Suppose $|f(z)| \to \infty$ as $z \to z_0$. Then there are a unique integer $k \geq 1$ and unique analytic function $g\colon D \mapsto \mathbb{C}$ such that $g(z_0) \neq 0$ and $f(z) = (z-z_0)^{-k} g(z)$ when $z \neq z_0$.*

Proof. Since $|f(z)| \to \infty$ as $z \to z_0$. Then we can find $\delta > 0$ such that $z \in B_\delta(z_0) \Rightarrow |f(z)| \geq 1$. Let

$$h(z) = \begin{cases} \frac{1}{f(z)} & 0 < |z-z_0| < \delta \\ 0 & z = z_0. \end{cases}$$

As $\frac{1}{f(z)} \to 0$ as $z \to z_0$, h is analytic in $B_\delta(z_0)$. Then $\exists\, k$ and $l\colon B_\delta(z_0) \mapsto \mathbb{C}$ such that l is analytic and $h(z) = (z - z_0)^k l(z)$ if $z \in B_\delta(z_0)$, $l(z_0) \neq 0$. Since l is continuous, we can find $0 < \delta_1 \leq \delta$ such that $l(z) \neq 0$ if $z \in B_{\delta_1}(z_0)$. Now let

$$g(z) = \begin{cases} \frac{1}{l(z)} & 0 \leq |z - z_0| < \delta_1 \\ (z - z_0)^k f(z) & z \neq z_0. \end{cases}$$

The definitions agree and g has the required properties. Uniqueness follows as before. □

6.2 Classification of Isolated Singularities

Definition 6.11. *Let D be a domain, $z_0 \in D$ and $f\colon D \setminus \{z_0\} \mapsto \mathbb{C}$ be analytic. Then z_0 is a singularity of f.*

- *If f is bounded in a neighbourhood of z_0 the singularity is called removable as we can define an analytic function $g\colon D \mapsto \mathbb{C}$ such that f and g agree on $D \setminus \{z_0\}$.*
- *If $|f(z)| \to \infty$ as $z \to z_0$ and if k is the integer from previous proposition the singularity is a pole of order k.*
- *All other singularities are called essential.*

Theorem 6.12 (Casorati-Weierstrass Theorem). *Let D be a domain, $z_0 \in D$ and $f\colon D \setminus \{z_0\} \mapsto \mathbb{C}$ be analytic with an essential singularity at z_0. Then for every $w \in \mathbb{C}$, $\exists\, z_n \to z_0$ such that $f(z_n) \to w$.*

Proof. Suppose otherwise. Then we can find $w \in \mathbb{C}$ such that $0 < |z - z_0| < \delta \Rightarrow |f(z) - w| \geq \epsilon$. Then $g(z) = 1/(f(z) - w)$ is analytic and bounded in $\{z : 0 < |z - z_0| < \delta\}$. Then g has a removable singularity at z_0 (in $B_\delta(z_0)$), so we can find an analytic function $h\colon B_\delta(z_0) \mapsto \mathbb{C}$ such that $h(z) = g(z)$ when $z \neq z_0$. Thus, when $z \neq z_0$, $f(z) = w + \frac{1}{h(z)}$ and so f has either a pole or removable singularity at z_0. □

Theorem 6.13 (Laurent's Theorem). [1] *Let D be the (non-empty) domain $\{z : a < |z - z_0| < b\}$ and let $f\colon D \mapsto \mathbb{C}$ be analytic. Then there exist unique coefficients $(a_n)_{n \in \mathbb{Z}}$ such that*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \;\forall\, z \in D.$$

Proof. Pick r and ρ such that $a < r < |z - z_0| < \rho < b$. Let γ be the straight line path from $z_0 + r$ to $z_0 + \rho$. [2] It is not hard to see that the closed path $C_\rho(z_0) \vee -\gamma \vee$-$C_r(z_0) \vee \gamma$ is homotopic in $D \setminus \{z\}$ to a path of the form $C_\delta(z)$. Hence (by Cauchy's Integral Formula and homotopy invariance)

$$f(z) = \frac{1}{2\pi\imath} \int_{C_\rho(z_0)} \frac{f(w)}{w - z} \mathrm{d}w - \frac{1}{2\pi\imath} \int_{C_r(z_0)} \frac{f(w)}{w - z} \mathrm{d}w.$$

[1] This is not strictly in the schedules, but is covered in Complex Methods. The proof that follows is slightly sketchy.

[2] Unless this goes through z, in which case take a small detour about z. I told you it was sketchy.

Just as in the proof of Taylor's theorem, expand in binomial series to get

$$\frac{1}{2\pi\imath}\int_{C_\rho(z_0)}\frac{f(w)}{w-z}\mathrm{d}w = \sum_{n=0}^{\infty} a_n(z-z_0)^n$$
$$a_n = \frac{1}{2\pi\imath}\int_{C_\rho(z_0)}\frac{f(w)}{(w-z_0)^{n+1}}\mathrm{d}w$$

and

$$-\frac{1}{2\pi\imath}\int_{C_r(z_0)}\frac{f(w)}{w-z}\mathrm{d}w = \frac{1}{2\pi\imath}\int_{C_r(z_0)}\frac{f(w)}{z-z_0-(w-z_0)}\mathrm{d}w$$

as before,

$$= \sum_{n=-\infty}^{-1} a_n(z-z_0)^n$$
$$a_n = \frac{1}{2\pi\imath}\int_{C_r(z_0)}\frac{f(w)}{(w-z_0)^{n+1}}\mathrm{d}w$$

or, using homotopy invariance

$$a_n = \frac{1}{2\pi\imath}\int_{C_\rho(z_0)}\frac{f(w)}{(w-z_0)^{n+1}}\mathrm{d}w.$$

For uniqueness, note that

$$\frac{1}{2\pi\imath}\int_{C_\rho(z_0)}\frac{f(z)}{(w-z_0)^{k+1}}\mathrm{d}w = \sum_{n=-\infty}^{\infty}\frac{a_n}{2\pi\imath}\int_{C_\rho(z_0)}(w-z_0)^{n-k-1}\,\mathrm{d}w$$
$$= a_k.$$

□

Let D be a domain, $z_0 \in D$, $f\colon D \setminus \{z_0\} \mapsto \mathbb{C}$ be analytic. Pick R such that $B_R(z_0) \subset D$. Then f has a Laurent expansion $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ in $\{z : 0 < |z-z_0| < R\}$. Let $k = \inf\{n : a_n \neq 0\}$. Then if $k \geq 0$, f has a removable singularity at z_0 and if k is finite but negative, f has a pole of order $-k$ at z_0. If k is not finite, then f has an essential singularity at z_0. The converse is also clear.

Theorem 6.14 (Maximum Modulus Theorem). *Let D be a domain and $f\colon D \mapsto \mathbb{C}$ be analytic. Suppose $|f|$ has a local maximum. Then f is constant.*

Proof. Suppose $z_0 \in D$ and δ are such that $|f(z_0)| \geq |f(z)|$ whenever $z \in \overline{B_\delta(z_0)}$. Then Cauchy's integral formula implies

$$f(z_0) = \frac{1}{2\pi\imath}\int_{C_\delta(z_0)}\frac{f(z)}{z-z_0}\mathrm{d}z.$$

Now pick θ such that $|f(z_0)| = e^{\imath\theta} f(z_0)$.

$$\begin{aligned}
\left| \frac{1}{2\pi\imath} \int_{C_\delta(z_0)} \frac{f(z)}{z - z_0} \mathrm{d}z \right| &= \frac{1}{2\pi\imath} \int_{C_\delta(z_0)} \frac{e^{\imath\theta} f(z)}{z - z_0} \mathrm{d}z \\
&= \frac{1}{2\pi\imath} \int_0^{2\pi} \frac{e^{\imath\theta} f(z_0 + \delta e^{\imath\phi})}{\delta e^{\imath\phi}} \imath\delta e^{\imath\phi} \mathrm{d}\phi \\
&= \frac{1}{2\pi} \int_0^{2\pi} \Re\left(e^{\imath\theta} f(z_0 + \delta e^{\imath\phi}) \right) \mathrm{d}\phi \\
&\leq \frac{1}{2\pi} \int_0^{2\pi} \left| e^{\imath\theta} f(z_0 + \delta e^{\imath\phi}) \right| \mathrm{d}\phi \\
&\leq |f(z_0)| \,.
\end{aligned}$$

But we know that equality occurs, so $e^{\imath\theta} f(z_0) = e^{\imath\theta} f(z_0 + \delta e^{\imath\phi})$ $\forall\phi$ (using a result of real analysis). So there exist non-isolated z where $f(z) = f(z_0)$, hence f is constant. □

Chapter 7

Winding Numbers

7.1 Introduction and Definition

Definition 7.1. *Let* $z \in \mathbb{C} \setminus \{0\}$*. A value of* $\log z$ *is a complex number* w *such that* $e^w = z$*. If* $w = a + \imath b$ *is a value of* $\log z$*, then* $a = \log |z|$*. It is clear that* $a + \imath b$ *is a value of* $\log z$ *iff* $a + \imath(b + 2n\pi)$ *is a value of* $\log z$ $\forall\, n \in \mathbb{Z}$*.*

Definition 7.2. *Let* $z \in \mathbb{C} \setminus \{0\}$*. A value of* $\arg z$ *is a real number* θ *such that* $z = |z|\, e^{\imath\theta}$*.* θ *is a value of* $\arg z$ *iff* $\theta + 2n\pi$ *is a value of* $\arg z$ $\forall\, n \in \mathbb{Z}$ *iff* $\log |z| + \imath\theta$ *is a value of* $\log z$*.*

Definition 7.3. *The principal value of* $\begin{matrix}\log z \\ \arg z\end{matrix}$ *is* $\begin{matrix}\log |z| + \imath\theta \\ \theta\end{matrix}$ *such that* $-\pi < \theta \leq \pi$*.*

Definition 7.4. *Let* D *be a domain such that* $0 \notin D$*. A continuous branch of* $\begin{matrix}\log z \\ \arg z\end{matrix}$ *is a continuous function* $f \colon D \mapsto \mathbb{C}$ *such that* $f(z)$ *is a value of* $\begin{matrix}\log z \\ \arg z\end{matrix}$ $\forall\, z \in D$*. This need not exist (for instance if* $D = \mathbb{C} \setminus \{0\}$*).*

Before doing anything with this, a lemma is useful.

Lemma 7.5. *Let* D *be a simply connected domain and* $f \colon D \mapsto \mathbb{C}$ *be analytic. Then* f *has an antiderivative.*

Proof. Take $z_0 \in D$ (D tacitly assumed to be non-empty) and define F by setting

$$F(z) = \int_\phi f(w)\mathrm{d}w, \text{ where } \phi \text{ is some path from } z_0 \text{ to } z.$$

By Cauchy's Theorem, this is well-defined and the proof that F is an antiderivative of f is more or less identical to the proof that $\int_{\partial T} f(w)\mathrm{d}w = 0$ for all triangles T implies that f has an antiderivative in a star-shaped domain. □

Lemma 7.6. *Let* D *be a simply connected domain not containing* 0*. Let* z_0 *be in* D *and* w_0 *be a value of* $\log z_0$*. Then there is a unique continuous branch* L *of* $\log$ *on* D *such that* $L(z_0) = w_0$*.*

Proof. $f(z) = \frac{1}{z}$ is analytic on D so by above lemma has an antiderivative L. By adding a suitable constant we may assume that $L(z_0) = w_0$. Now consider $g(z) = ze^{-L(z)}$. $g'(z) = e^{-L(z)}(1 - zL'(z)) = 0$ so g is constant on D. Now $g(z_0) = 1$, so L is a continuous (and even analytic) branch of $\log z$. If L_1 is another continuous branch, then $\frac{L_1(z)-L(z)}{2\pi\imath}$ is an integer for all $z \in D$. But L_1 and L are continuous, so this must be constant and as $L(z_0) = L_1(z_0)$, $L \equiv L_1$. □

Definition 7.7. *Let D be a simply connected domain, take $z_0 \in \mathbb{C} \setminus D$ and consider a path $\phi\colon [a,b] \mapsto D$. The change (or variation) in $\log(z - z_0)$is defined as $L(\phi(b)) - L(\phi(a))$ where L is any continuous branch of $\log(z - z_0)$ on D. Note that this is well defined, and from the way that L was produced, is also equal to*

$$\int_\phi \frac{\mathrm{d}z}{z - z_0}.$$

Now let D be any domain, $z_0 \in D$ and $\phi\colon [a,b] \mapsto D$ be a path such that $z_0 \notin \phi^*$. Since ϕ^* is compact and $(\mathbb{C} \setminus D) \cup \{z_0\}$ is closed, there exists $\epsilon > 0$ such that $B_\epsilon(z) \subset D$ and $z_0 \notin B_\epsilon(z)$ for all $z \in \phi^*$.

Also ϕ is uniformly continuous, so $\exists\, n \in \mathbb{N}$ such that

$$|x - y| \leq \frac{1}{n} \Rightarrow |\phi(x) - \phi(y)| < \epsilon.$$

Now let $x_i = a + \frac{i}{n}(b - a)$ for $i = 0, 1, \ldots, n$ and let ϕ_i be the restriction of ϕ to $[x_{i-1}, x_i]$ and $C_i = B_\epsilon(\phi(x_i))$. Then $\phi_i^* \subset C_i$.

Definition 7.8. *The change in $\log(z - z_0)$ along ϕ is defined to be the sum of the changes in $\log$ for each of the ϕ_i. This is not circular – the C_i's are manifestly simply connected and do not contain z_0. Thus the change in $\log$ is also equal to*

$$\int_\phi \frac{\mathrm{d}z}{z - z_0}.$$

When we chose continuous branches L_i of $\log(z - z_0)$ in each C_i, we could, by adding suitable constants, ensure that $L_i(\phi(x_i)) = L_{i+1}(\phi(x_i))$. If we do that, then the change along ϕ is

$$\sum_{i=1}^{n} L_i(\phi(x_i)) - L_i(\phi(x_{i-1})) = L_n(\phi(b)) - L_1(\phi(a))$$

If ϕ is closed, this must be $2\pi k\imath$ for some $k \in \mathbb{Z}$.

Definition 7.9. *The winding number of a closed path ϕ about z_0 is defined as this k. It is denoted as $w(\phi, z_0)$ and is equal to*

$$\frac{1}{2\pi\imath}\int_\phi \frac{\mathrm{d}z}{z - z_0}.$$

From the above formula, we see that for $z_0 \notin \phi^*$, $w(\phi, z_0)$ is an analytic function of z_0 with derivative

$$\frac{1}{2\pi\imath}\int_\phi \frac{\mathrm{d}z}{(z - z_0)^2}.$$

It is therefore continuous, and as it only takes integer values must be constant on components of $\mathbb{C} \setminus \phi^*$. Since ϕ^* is compact, there is a unique unbounded component of $\mathbb{C} \setminus \phi^*$ where the winding number is zero.

To see this, let $|z_0| > 2\max\{|z| : z \in \phi^*\}$. Then

$$w(\phi, z_0) = \frac{1}{2\pi}\left|\int_\phi \frac{\mathrm{d}z}{z - z_0}\right| \leq \frac{L(\phi)}{\pi\,|z_0|} \to 0 \text{ as } |z_0| \to \infty.$$

7.2 Residues

Let D be a domain and f be a function analytic on D except at finitely many points $z_1, \ldots, z_k$. Given $z \in D$, we can find $\delta > 0$ such that $B_\delta(z)$ contains none of the z_i unless $z = z_i$, in which case $B_\delta(z) \cap \{z_1, \ldots, z_k\} = \{z_i\}$. Inside $B_\delta(z)$, f has a Laurent expansion

$$f(w) = \sum_{n=-\infty}^{\infty} a_n (w - z)^n.$$

Definition 7.10. *The residue of f at z is defined as a_{-1} and is written $\operatorname{Res}(f, z)$.*

If $z \notin \{z_1, \ldots, z_k\}$, then $\operatorname{Res}(f, z) = 0$. Now, at z_i, write

$$f(z) = \sum_{n=-\infty}^{\infty} a_n^{(i)} (z - z_i)^n.$$

Then

$$\operatorname{Res}(f, z_i) = \frac{1}{2\pi\imath}\int_{C_\delta(z_i)} f(z)\mathrm{d}z.$$

This gives an alternative definition of $\operatorname{Res}(f, z_i)$ not involving Laurent expansions.

Definition 7.11. *The principal part of f at z_i is defined to be the function*

$$g_i(z) = \sum_{n=-\infty}^{-1} a_n^{(i)} (z - z_i)^n.$$

g is analytic on $D \setminus \{z_i\}$ and $f - g_i$ has a removable singularity at z_i.

Theorem 7.12 (Cauchy's Residue Theorem). *Let D be a simply connected domain and f and $z_1, \ldots, z_k$ be as above. Let ϕ be a closed path in D such that $\phi^* \cap \{z_1, \ldots, z_k\} = \emptyset$. Then*

$$\int_\phi f(z)\mathrm{d}z = 2\pi\imath \sum_{j=1}^{k} \operatorname{Res}(f, z_i) w(\phi, z_i).$$

Proof. $f - (g_1 + \cdots + g_k)$[1] is analytic on D except for removable singularities at $z_1, \ldots, z_k$. Let $h\colon D \mapsto \mathbb{C}$ be analytic such that $h(z) = f(z) - (g_1(z) + \cdots +$

[1]The g_i's are the relevant principal parts.

$g_k(z))\ \forall z \in D \setminus \{z_1, \ldots, z_k\}$. Then by Cauchy's theorem,

$$\int_\phi h(z)\mathrm{d}z = 0 \quad \text{and hence}$$

$$\int_\phi f(z)\mathrm{d}z = \sum_{j=1}^{k} \int_\phi g_j(z)\mathrm{d}z$$
$$= 2\pi\imath \sum_{j=1}^{k} \mathrm{Res}(f, z_i) w(\phi, z_i).$$

□

Definition 7.13. *Let D be a domain, $z_0 \in D$ and let f be a function analytic when $0 < |z - z_0| < \epsilon$ for some $\epsilon > 0$. Recall that if f has a removable singularity or pole at z_0 we can write $f(z) = (z - z_0)^k g(z)$ where k and g are uniquely determined, $g(z_0) \neq 0$ and g analytic. Then the integer k is called the order of f at z_0 and is written* $\mathrm{ord}(f, z_0)$.

Theorem 7.14. *Let D be a domain and let $f : D \mapsto \mathbb{C}$ be analytic except at finitely many poles. Suppose also that f has finitely many zeros in D, and let the zeros and poles be $z_1, \ldots, z_k$. Let ϕ be a closed path in D such that $\phi^* \cap \{z_1, \ldots, z_k\} = \emptyset$. Then*

$$\frac{1}{2\pi\imath} \int_\phi \frac{f'(z)}{f(z)}\mathrm{d}z = \sum_{j=1}^{k} \mathrm{ord}(f, z_j) w(\phi, z_j).$$

Proof. By the residue theorem

$$\frac{1}{2\pi\imath} \int_\phi \frac{f'(z)}{f(z)}\mathrm{d}z = \sum_{j=1}^{k} \mathrm{Res}(\frac{f'}{f}, z_j) w(\phi, z_j).$$

Near z_j, $f = (z - z_j)^r g(z)$ with $g(z_j) \neq 0$ and $r = \mathrm{ord}(f, z_j)$. Then

$$\frac{f'(z)}{f(z)} = \frac{r}{z - z_j} + \frac{g'(z)}{g(z)}$$

and $\mathrm{Res}(\frac{f'}{f}, z_j) = \mathrm{ord}(f, z_j)$. Summing over j gives the result. □

Notation. *Write $ZP(f, \phi)$ for* $\frac{1}{2\pi\imath} \int_\phi \frac{f'(z)}{f(z)}\mathrm{d}z$.

N.B.

$$\frac{1}{2\pi\imath} \int_\phi \frac{f'(z)}{f(z)}\mathrm{d}z = \frac{1}{2\pi\imath} \int_{f \circ \phi} \frac{\mathrm{d}z}{z} = w(f \circ \phi, 0).$$

Theorem 7.15 (Rouché's Theorem). *Let D be a domain, ϕ be a closed path in D and f and g be functions with the following properties:*

1. *f and g are analytic on D except for finitely many poles, none of which lie on ϕ^*.*
2. *f and $f + g$ have finitely many zeros, none of which lie on ϕ^*.*

3. $|g(z)| < |f(z)| \ \forall\, z \in \phi^*$.

Then $ZP(f+g, \phi) = ZP(f, \phi)$.

Proof. It follows from the definition of order at a point that $\operatorname{ord}(f+g, z) = \operatorname{ord}(f, z) + \operatorname{ord}(\frac{f+g}{f}, z)$. Hence, putting $h(z) = 1 + \frac{g(z)}{f(z)}$, we have $ZP(f+g, \phi) = ZP(f, \phi) + ZP(h, \phi)$. But $ZP(h, \phi) = w(h \circ \phi, 0)$, and for $z \in \phi^*$, $\Re h(z) \geq 1 - \left|\frac{g(z)}{f(z)}\right| > 0$. But there is a continuous branch of $\log$ on the right half-plane, so $w(h \circ \phi, 0) = 0$. □

Theorem 7.16 (Local Mapping Theorem). *Let D be a domain, $z_0 \in D$ and $f\colon D \mapsto \mathbb{C}$ be analytic and non-constant. Then for $\epsilon > 0$ sufficiently small, there exists $\delta > 0$ such that whenever $0 < |w - w_0| < \delta$, there are exactly k values of z such that $0 < |z - z_0| < \epsilon$ and $f(z) = w$, where $k = \operatorname{ord}(f - w_0, z_0)$.*

Proof. Choose $\epsilon > 0$ small enough such that whenever $0 < |z - z_0| < 2\epsilon$,

1. $f(z) \neq w_0$,
2. $f'(z) \neq 0$,
3. $z \in D$.

Note that 1 and 2 are possible by the Identity Theorem. Now $C_\epsilon(z_0)^*$ is compact, so put $\delta = \inf\{|f(z) - w_0| : z \in C_\epsilon(z_0)^*\} > 0$.

Then $\forall\, z \in C_\epsilon(z_0)^*$, $|w - w_0| < |f(z) - w_0|$. Hence by Rouché's Theorem,

$$\begin{aligned} k &= \text{number of zeros up to multiplicity of } f(z) - w_0 \text{ in } B_\epsilon(z_0) \\ &= \text{number of zeros of } f(z) - w_0 \quad (= f(z) - w_0 - (w - w_0)). \end{aligned}$$

But every zero of $f(z) - w$ is simple, since $f' \neq 0$. □

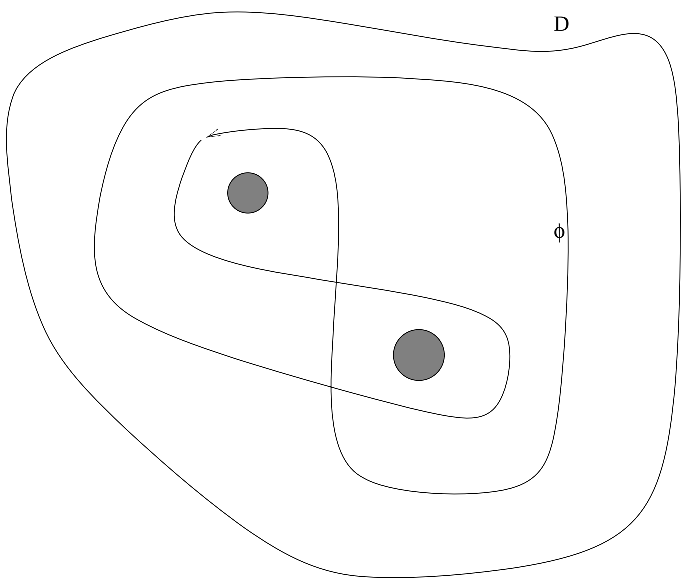
Corollary 7.17 (Open Mapping Theorem). *Let D be a domain and $f\colon D \mapsto \mathbb{C}$ be analytic and non-constant. Then if $U \subset D$ is open, $f(U)$ is open.*

Proof. Let $w_0 \in f(U)$ and z_0 be such that $f(z_0) = w_0$. The Local Mapping Theorem provides $\delta > 0$ such that $B_\delta(w_0) \subset f(B_\epsilon(z_0)) \subset f(U)$. Hence $f(U)$ is open. □

Remark. *The Maximum Modulus Theorem follows immediately.*

Chapter 8

Cauchy's Theorem (homology version)



Let D be the domain and ϕ the path shown. It can be shown by methods of algebraic topology that ϕ is not contractible in D. However, it is also clear that $\int_\phi f(z)\mathrm{d}z = 0$ for all analytic functions f on D. We shall ask which paths have this property. This section of the course is starred.

A chain in a domain D is a finite sequence $(\phi_1, \ldots, \phi_N)$ of paths. Two chains $(\phi_1, \ldots, \phi_N)$ and $(\psi_1, \ldots, \psi_N)$ are directly equivalent if there is a permutation π of the set $\{1, 2, \ldots, N\}$ such that $\psi_i = \phi_{\pi(i)}$ for every i. A subdivision of a chain $(\phi_1, \ldots, \phi_N)$ is a chain

$$(\phi_{11}, \phi_{12}, \ldots, \phi_{1M_1}, \phi_{21}, \ldots, \phi_{2M_2}, \ldots, \phi_{N1}, \ldots, \phi_{NM_N})$$

such that $\phi_i = \phi_{i1} \vee \phi_{i2} \vee \cdots \vee \phi_{iM_i}$ for every i. Two chains are equivalent if they have directly equivalent subdivisions.

A cycle is a chain $(\phi_1,\dots,\phi_N)$ such that each ϕ_i is a closed path[1]. Two cycles $(\phi_1,\dots,\phi_N)$ and $(\psi_1,\dots,\psi_N)$ are homotopic if ϕ_i is homotopic ψ_i for every i. Two cycles $\Phi=(\phi_1,\dots,\phi_N)$ and $\Psi=(\psi_1,\dots,\psi_N)$ are homologous if there is a sequence $\Phi=\Phi_0,\Phi_1,\dots,\Phi_K=\Psi$ such that for every i, Φ_{i-1} and Φ_i are either equivalent or homotopic.

If $\Phi=(\phi_1,\dots,\phi_N)$ is a chain in a domain D and $f\colon D\mapsto\mathbb{C}$ is continuous, then $\int_\Phi f(z)\mathrm{d}z$ is defined to be $\sum_{i=1}^N\int_{\phi_i}f(z)\mathrm{d}z$. If Φ is also a cycle, and $z_0\in\mathbb{C}\setminus\Phi^*$, then the winding number of Φ about z_0 is defined to be

$$w(\Phi,z_0)=\sum_{i=1}^N w(\phi_i,z_0)=\frac{1}{2\pi\imath}\int_\Phi\frac{\mathrm{d}z}{z-z_0}.$$

Theorem 8.1. *Let D be a domain and Φ be a cycle homologous in D to a point. Then $\int_\Phi f(z)\mathrm{d}z=0$ for every analytic function $f\colon D\mapsto\mathbb{C}$.*

Proof. Trivial consequence of homotopy invariance. □

This certainly deals with the path shown above. The converse of this theorem is also true, but somewhat harder to prove. The main result of this section is a characterization in terms of winding numbers of those cycles for which the integral of any analytic function vanishes.

Theorem 8.2. *Let D be a domain and let Φ be a cycle in D with $w(\Phi,z_0)=0$ for every complex number $z_0\notin D$. Then $\int_\Phi f(z)\mathrm{d}z=0$ for every analytic function $f\colon D\mapsto\mathbb{C}$.*

The converse of this theorem is obvious, using the function $f(z)=(z-z_0)^{-1}$. In order to prove this theorem we need another set of definitions from algebraic topology and 3 easy lemmas.

Given a real number $\delta>0$, we define $\Sigma(\delta)$ to be the set of all squares $S\subset\mathbb{C}$ of the form

$$\{z\in\mathbb{C}:m\delta\le\Re z\le(m+1)\delta,n\delta\le\Im z\le(n+1)\delta\}$$

where m and n are integers. Given such a square S, we denote by ∂S the boundary of S, oriented anticlockwise. A square complex of mesh δ is a subset $\Sigma\subset\Sigma(\delta)$. If $\Sigma=\{S_1,\dots,S_N\}$ is a square complex, then an edge of one of the S_i is called internal if it is shared by some other S_j, and is otherwise called external. The boundary $\partial\Sigma$ of Σ is defined to be the chain of all external edges of Σ (with their directions coming from the orientations of the relevant ∂S_i). We write Σ^* for the union of the squares that make up Σ (so that $\Sigma^*\subset\mathbb{C}$ and $\Sigma\subset P[\mathbb{C}]$).

Lemma 8.3. *Let Σ be a square complex. Then $\partial\Sigma$ is equivalent to a cycle.*

Proof. We can form a directed graph, where the vertices are all points of the form $\delta(m+n\imath)$ and the edges are the external edges of Σ (with their directions). It does not take long to check that at any vertex the number of edges going in equals the number of edges coming out. Now start at a vertex v which has at least one vertex which has at least one edge coming out of it, and move along edges in the forward direction for as long as possible without repeating an edge. As there are finitely many edges this process must stop, and because of the condition just mentioned must stop at v. The result is a closed path. If we remove this path, we obtain a directed graph with fewer edges satisfying the same condition, so by induction the result is proved. □

[1] It is more usual to define a cycle to be a chain equivalent to what I have called a cycle.

Lemma 8.4. *Let* $\Sigma = \{S_1, \dots, S_N\}$ *be a square complex and let* D *be a domain containing* Σ^**. Let* $f\colon D \mapsto \mathbb{C}$ *be continuous. Then* $\int_{\partial\Sigma} f(z)\mathrm{d}z = \sum_{i=1}^{N} \int_{\partial S_i} f(z)\mathrm{d}z$.

Proof. The contributions to the right-hand side from integrals along internal edges cancel, leaving only the integrals along external edges, which by definition is the left-hand side. □

Lemma 8.5. *Let* $\Sigma = \{S_1, \dots, S_N\}$ *be a square complex and let* $z_0 \in \operatorname{int}(\Sigma^*)$*. Let* f *be a function analytic on a domain that includes* Σ^**. Then* $f(z_0) = \frac{1}{2\pi\imath} \int_{\partial\Sigma} \frac{f(z)}{z - z_0}\mathrm{d}z$.

Proof. Firstly, suppose z_0 lies in the interior of S_i for some i. When $j \neq i$, Cauchy's theorem implies that $\int_{\partial S_j} \frac{f(z)}{z-z_0}\mathrm{d}z$. Also, ∂S_i is homotopic to $C_\delta(z_0)$ for some $\delta > 0$, so by Cauchy's integral formula $f(z_0) = \frac{1}{2\pi\imath} \int_{\partial S_i} \frac{f(z)}{z-z_0}\mathrm{d}z$. The result then follows from above lemma.

Now if z_0 lies on an internal edge the result follows by continuity. □

Proof of theorem. Let $X \subset D$ be $\mathbb{C} \setminus \{z : w(\Phi, z) = 0\} = \{z : w(\Phi, z) \neq 0\} \cup \Phi^*$. Since $w(\Phi, z)$ is continuous on the open set $\mathbb{C} \setminus \Phi^*$, we see that $\{z : w(\Phi, z) = 0\}$ is open, so that X is closed. Also, since $w(\Phi, z) = 0$ on the unbounded component of $\mathbb{C} \setminus \Phi^*$, X is bounded. Thus X is compact, from which it follows that we can choose $\delta > 0$ such that $B_{2\delta}(z) \subset D$ whenever $z \in X$.

Define a square complex Σ of mesh δ by taking every square $S \in \Sigma(\delta)$ such that $S \cap C \neq \emptyset$. Note that S is included even if S intersects X only on the boundary - this is important. It is clear that $X \subset \Sigma^*$, but we also have $X \subset \operatorname{int}\Sigma^*$ as the definition of Σ does not allow $z \in X$ to be on an external edge. Also, by our choice of δ, we have $\Sigma^* \subset D$.

Hence, if $z \in \Phi^* \subset X$, above lemma gives that $f(z) = \frac{1}{2\pi\imath} \int_{\partial\Sigma} \frac{f(w)}{w-z}\mathrm{d}w$. It follows that

$$\begin{aligned}
\int_\Phi f(z)\mathrm{d}z &= \frac{1}{2\pi\imath} \int_\Phi \int_{\partial\Sigma} \frac{f(w)}{w - z}\mathrm{d}w\mathrm{d}z \\
&= \int_{\partial\Sigma} f(w) \frac{1}{2\pi\imath} \int_\Phi \frac{\mathrm{d}z}{w - z}\mathrm{d}w \\
&= -\int_{\partial\Sigma} f(w) w(\Phi, w)\mathrm{d}w.
\end{aligned}$$

The above change of integrals will not be justified, but we are talking about continuous functions on closed, bounded subsets of $\mathbb{R}$, where the justification is relatively easy. Now since $X \cap \partial\Sigma = \emptyset$, we have $w(\Phi, w) = 0$ for every $w \in \partial\Sigma$. This completes the proof. □

References

- G.J.O. Jameson, *A First Course on Complex Functions*, Chapman and Hall, out of print.

  This is a good book for the second half of the course — but probably not as good as Priestley.

- H.A. Priestley, *Introduction to Complex Analysis*, OUP, 1990.

  This book is highly recommended for the second half of the course.

- W.A. Sutherland, *Introduction to Metric and Topological Spaces*, OUP, 1975.

  This book is quite good for the first half of the course.

There are a lot of books on this course. Quality is variable.

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