

# 线性代数

习

题

解

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# 线性代数

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这是题文

这是题解

这是注释

习题一	.....	1
习题二	.....	13
习题三	.....	22
习题四	.....	31
习题五	.....	(缺)
习题六	.....	47

## 习题一

1.1 设  $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$ ,  $B = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & 4 \\ 0 & 5 & 1 \end{bmatrix}$

求  $3AB - 2A$  及  $A^T B$ .

$$\begin{aligned} 3AB - 2A &= 3 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & 4 \\ 0 & 5 & 1 \end{bmatrix} - 2 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \\ &= 3 \begin{bmatrix} 0 & 5 & 8 \\ 0 & -5 & 6 \\ 2 & 9 & 0 \end{bmatrix} - 2 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 15 & 24 \\ 0 & -15 & 18 \\ 6 & 27 & 0 \end{bmatrix} - \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & -2 \\ 2 & -2 & 2 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 13 & 22 \\ -2 & -17 & 20 \\ 4 & 29 & -2 \end{bmatrix} \end{aligned}$$

$$A^T B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & 4 \\ 0 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 5 & 8 \\ 0 & -5 & 6 \\ 2 & 9 & 0 \end{bmatrix}$$

1.2 计算下列乘积

(1)  $\begin{bmatrix} 4 & 3 & 1 \\ 1 & -2 & 3 \\ 5 & 7 & 0 \end{bmatrix} \begin{bmatrix} 7 \\ 2 \\ 1 \end{bmatrix}$

(2)  $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

(3)  $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \begin{bmatrix} -1 & 2 \end{bmatrix}$

(4)  $\begin{bmatrix} 2 & 1 & 4 & 0 \\ 1 & -1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 0 & -1 & 2 \\ 1 & -3 & 1 \\ 4 & 0 & -2 \end{bmatrix}$

(5)  $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

(1)  $\begin{bmatrix} 35 \\ 6 \\ 49 \end{bmatrix}$  (2)  $10$  (3)  $\begin{bmatrix} -2 & 4 \\ -1 & 2 \\ -3 & 6 \end{bmatrix}$  (4)  $\begin{bmatrix} 6 & -7 & 8 \\ 20 & -5 & -6 \end{bmatrix}$

(5)  $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{bmatrix}$   
 $= a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3$

1.3 设  $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ ,  $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ , 问下列各式是否成立?

(1)  $AB = BA$ ;

(2)  $(AB)^2 = A^2B^2$

(3)  $(A+B)^2 = A^2 + 2AB + B^2$ ;

(4)  $(A+B)(A-B) = A^2 - B^2$

(1)  $AB = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \neq BA = \begin{bmatrix} 0 & -1 \\ - & 0 \end{bmatrix}$

(2)  $(AB)^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \neq A^2B^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

(3)  $(A+B)^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \neq A^2 + 2AB + B^2 = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$

(4)  $(A+B)(A-B) = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \neq A^2 - B^2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

**1.4** 讨论下列命题是否正确:(1) 若  $A^2 = 0$ , 则  $A = 0$ ;(2) 若  $A^2 = A$ , 则  $A = 0$  或  $A = E$ ;(3) 若  $AB = AC$  且  $A \neq 0$ , 则  $B = C$ .(1) 不对. 反例:  $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ , 但  $A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ .(2) 不对. 反例: 设  $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ , 则  $A \neq 0$  且  $A \neq E$ , 但  $A^2 = A$ .(3) 不对. 反例: 设  $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ ,  $B = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$ ,  $C = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}$ , 则有  $AB = AC$  且  $A \neq 0$ , 但  $B \neq C$ .**1.5** 1.5 计算:

(1)  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n$ , (2)  $\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}^n$ , (3)  $\begin{pmatrix} 1 & & & \\ & 2 & & \\ & & 3 & \\ & & & 4 \end{pmatrix}^n$ .

(1)

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix},$$

...

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & n-1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$

(2)

$$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}^2 = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda^2 & 2\lambda & 1 \\ 0 & \lambda^2 & 2\lambda \\ 0 & 0 & \lambda^2 \end{pmatrix},$$

$$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}^3 = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \begin{pmatrix} \lambda^2 & 2\lambda & 1 \\ 0 & \lambda^2 & 2\lambda \\ 0 & 0 & \lambda^2 \end{pmatrix} = \begin{pmatrix} \lambda^3 & 3\lambda^2 & 3\lambda \\ 0 & \lambda^3 & 3\lambda^2 \\ 0 & 0 & \lambda^3 \end{pmatrix},$$

$$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}^4 = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \begin{pmatrix} \lambda^3 & 3\lambda^2 & 3\lambda \\ 0 & \lambda^3 & 3\lambda^2 \\ 0 & 0 & \lambda^3 \end{pmatrix} = \begin{pmatrix} \lambda^4 & 4\lambda^3 & 6\lambda \\ 0 & \lambda^4 & 4\lambda^2 \\ 0 & 0 & \lambda^4 \end{pmatrix},$$

...

$$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}^n = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \begin{pmatrix} \lambda^{n-1} & (n-1)\lambda^{n-2} & \frac{(n-1)(n-2)}{2}\lambda^{n-3} \\ 0 & \lambda^{n-1} & (n-1)\lambda^{n-2} \\ 0 & 0 & \lambda^{n-1} \end{pmatrix} = \begin{pmatrix} \lambda^n & n\lambda^{n-1} & \frac{n(n-1)}{2}\lambda^{n-2} \\ 0 & \lambda^n & (n-1)\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{pmatrix}.$$

$$\begin{pmatrix} 1 & & & \\ & 2 & & \\ & & 3 & \\ & & & 4 \end{pmatrix}^2 = \begin{pmatrix} 1 & & & \\ & 2 & & \\ & & 3 & \\ & & & 4 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 2 & & \\ & & 3 & \\ & & & 4 \end{pmatrix} = \begin{pmatrix} 1^2 & & & \\ & 2^2 & & \\ & & 3^2 & \\ & & & 4^2 \end{pmatrix},$$

(3) ...

$$\begin{pmatrix} 1 & & & \\ & 2 & & \\ & & 3 & \\ & & & 4 \end{pmatrix}^n = \begin{pmatrix} 1 & & & \\ & 2 & & \\ & & 3 & \\ & & & 4 \end{pmatrix} \begin{pmatrix} 1^{n-1} & & & \\ & 2^{n-1} & & \\ & & 3^{n-1} & \\ & & & 4^{n-1} \end{pmatrix} = \begin{pmatrix} 1^n & & & \\ & 2^n & & \\ & & 3^n & \\ & & & 4^n \end{pmatrix}.$$

**1.6** 1.6 设方阵  $A$  满足矩阵方程  $A^2 - A - 2E = 0$ , 证明  $A$  及  $A+2E$  都可逆, 并求  $A^{-1}$  及  $(A+2E)^{-1}$ .

由  $A^2 - A - 2E = 0$  得  $\frac{1}{2}(A-E)A = E$ , 故  $A$  可逆, 且  $A^{-1} = \frac{1}{2}(A-E)$ .

由  $A^2 - A - 2E = 0$  也可得  $(A+2E)(A-3E) = -4E$  或  $(A+2E)\left[-\frac{1}{4}(A-3E)\right] = E$ , 故  $A+2E$  可逆, 且

$$(A+2E)^{-1} = -\frac{1}{4}(A-3E).$$

**1.7** 1.7 利用初等行变换求下列矩阵的逆矩阵:

$$(1) \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

$$(2) \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$$

$$(3) \begin{bmatrix} 1 & -2 & 2 \\ 2 & -3 & 6 \\ 1 & 1 & 7 \end{bmatrix}$$

$$(4) \begin{bmatrix} 1 & 2 & -4 \\ -1 & -1 & 5 \\ 2 & 7 & -3 \end{bmatrix}$$

$$(5) \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$

$$(6) \begin{bmatrix} 3 & 5 & 0 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 2 & 0 & 2 \end{bmatrix}$$

$$(1) \begin{pmatrix} 1 & 2 & 2 & | & 1 & 0 & 0 \\ 2 & 1 & -2 & | & 0 & 1 & 0 \\ 2 & -2 & 1 & | & 0 & 0 & 1 \end{pmatrix} \xrightarrow[r_2-r_1]{r_3-2r_2} \begin{pmatrix} 1 & 2 & 2 & | & 1 & 0 & 0 \\ 0 & -3 & -6 & | & -2 & 1 & 0 \\ 0 & -3 & 3 & | & 0 & -1 & 1 \end{pmatrix} \xrightarrow[r_1-2r_2]{r_3-r_2} \begin{pmatrix} 1 & 0 & -2 & | & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 1 & 2 & | & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 9 & | & 2 & -2 & 1 \end{pmatrix} \xrightarrow[r_1+2r_3]{\frac{1}{9}r_3} \begin{pmatrix} 1 & 0 & 0 & | & \frac{1}{9} & \frac{2}{9} & \frac{2}{9} \\ 0 & 1 & 0 & | & \frac{2}{9} & \frac{1}{9} & -\frac{2}{9} \\ 0 & 0 & 1 & | & \frac{2}{9} & -\frac{2}{9} & \frac{1}{9} \end{pmatrix},$$

$$\therefore \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{9} & \frac{2}{9} & \frac{2}{9} \\ \frac{2}{9} & \frac{1}{9} & -\frac{2}{9} \\ \frac{2}{9} & -\frac{2}{9} & \frac{1}{9} \end{pmatrix}$$

$$(2) \begin{pmatrix} 1 & 0 & 2 & | & 1 & 0 & 0 \\ 2 & -1 & 3 & | & 0 & 1 & 0 \\ 4 & 1 & 8 & | & 0 & 0 & 1 \end{pmatrix} \xrightarrow[r_3-4r_1]{r_2-2r_1} \begin{pmatrix} 1 & 0 & 2 & | & 1 & 0 & 0 \\ 0 & -1 & -1 & | & -2 & 1 & 0 \\ 0 & 1 & 0 & | & -4 & 0 & 1 \end{pmatrix} \xrightarrow[r_1-2r_2]{r_2+r_3} \begin{pmatrix} 1 & 0 & 2 & | & 1 & 0 & 0 \\ 0 & 0 & -1 & | & -6 & 1 & 1 \\ 0 & 1 & 0 & | & -4 & 0 & 1 \end{pmatrix} \xrightarrow[r_1-2r_2]{-\frac{1}{3}r_2} \begin{pmatrix} 1 & 0 & 0 & | & -11 & 2 & 2 \\ 0 & 0 & 1 & | & 6 & -1 & -1 \\ 0 & 1 & 0 & | & -4 & 0 & 1 \end{pmatrix} \xrightarrow[r_2]{r_1+2r_2, -r_2} \begin{pmatrix} 1 & 0 & 0 & | & -11 & 2 & 2 \\ 0 & 1 & 0 & | & -4 & 0 & 1 \\ 0 & 0 & 1 & | & 6 & -1 & -1 \end{pmatrix},$$

$$\therefore \begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{pmatrix}^{-1} = \begin{pmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{pmatrix}$$

$$(3) \begin{pmatrix} 1 & -2 & 2 & | & 1 & 0 & 0 \\ 2 & -3 & 6 & | & 0 & 1 & 0 \\ 1 & 1 & 7 & | & 0 & 0 & 1 \end{pmatrix} \xrightarrow[r_3-2r_1]{r_3-r_2} \begin{pmatrix} 1 & -2 & 2 & | & 1 & 0 & 0 \\ 0 & 1 & 2 & | & -2 & 1 & 0 \\ 0 & 3 & 5 & | & -1 & 0 & 1 \end{pmatrix} \xrightarrow[r_3-3r_2]{r_1+2r_2} \begin{pmatrix} 1 & 0 & 6 & | & -3 & 2 & 0 \\ 0 & 1 & 2 & | & -2 & 1 & 0 \\ 0 & 0 & -1 & | & 5 & -3 & 1 \end{pmatrix} \xrightarrow[r_1+6r_3]{r_2+2r_3} \begin{pmatrix} 1 & 0 & 0 & | & 27 & -16 & 6 \\ 0 & 1 & 0 & | & 8 & -5 & 2 \\ 0 & 0 & -1 & | & 5 & -3 & 1 \end{pmatrix} \xrightarrow{-r_3} \begin{pmatrix} 1 & 0 & 0 & | & 27 & -16 & 6 \\ 0 & 1 & 0 & | & 8 & -5 & 2 \\ 0 & 0 & 1 & | & -5 & 3 & -1 \end{pmatrix},$$

$$\therefore \begin{pmatrix} 1 & -2 & 2 \\ 2 & -3 & 6 \\ 1 & 1 & 7 \end{pmatrix}^{-1} = \begin{pmatrix} 27 & -16 & 6 \\ 8 & -5 & 2 \\ -5 & 3 & -1 \end{pmatrix}$$

$$(4) \begin{pmatrix} 1 & 2 & -4 & | & 1 & 0 & 0 \\ -1 & -1 & 5 & | & 0 & 1 & 0 \\ 2 & 7 & -3 & | & 0 & 0 & 1 \end{pmatrix} \xrightarrow[r_3-2r_1]{r_2+r_1} \begin{pmatrix} 1 & 2 & -4 & | & 1 & 0 & 0 \\ 0 & 1 & 1 & | & 1 & 1 & 0 \\ 0 & 3 & 5 & | & -2 & 0 & 1 \end{pmatrix} \xrightarrow[r_3-3r_2]{r_1-2r_2} \begin{pmatrix} 1 & 0 & -6 & | & -1 & -2 & 0 \\ 0 & 1 & 1 & | & 1 & 1 & 0 \\ 0 & 0 & 2 & | & -5 & -3 & 1 \end{pmatrix} \xrightarrow[r_2-r_3]{r_1+3r_3, \frac{1}{2}r_3} \begin{pmatrix} 1 & 0 & 0 & | & -16 & -11 & 3 \\ 0 & 1 & 0 & | & \frac{7}{2} & \frac{5}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & | & -\frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & -4 \\ -1 & -1 & 5 \\ 2 & 7 & -3 \end{pmatrix}^{-1} = \begin{pmatrix} -16 & -11 & 3 \\ \frac{7}{2} & \frac{5}{2} & -\frac{1}{2} \\ -\frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{pmatrix}.$$

$$(5) \left( \begin{array}{cccc|cccc} 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 9 & 0 & 0 & 0 & 1 \end{array} \right) \xrightarrow{r_2+4r_3} \left( \begin{array}{cccc|cccc} 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 9 & 0 & 0 & 0 & 1 \end{array} \right) \xrightarrow{\begin{array}{l} \frac{1}{2}r_1 \\ -r_3 \\ \frac{1}{9}r_4 \end{array}} \left( \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{9} \end{array} \right)$$

$$\therefore \left( \begin{array}{cccc} 2 & 0 & 0 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 9 \end{array} \right)^{-1} = \left( \begin{array}{cccc} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \frac{1}{9} \end{array} \right)$$

$$(6) \left( \begin{array}{cccc|cccc} 3 & 5 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\ 1 & 2 & 0 & 2 & 0 & 0 & 0 & 1 \end{array} \right) \xrightarrow{r_1 \leftrightarrow r_3} \left( \begin{array}{cccc|cccc} 1 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 3 & 5 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 2 & 0 & 0 & 0 & 1 \end{array} \right) \xrightarrow{\begin{array}{l} r_3-3r_1 \\ r_4-r_1 \\ r_2-r_1 \end{array}} \left( \begin{array}{cccc|cccc} 1 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & -2 & 0 & 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & -5 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 2 & 0 & -1 & 0 & 1 \end{array} \right)$$

$$\xrightarrow{r_2 \leftrightarrow r_1} \left( \begin{array}{cccc|cccc} 1 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -5 & 1 & 0 & 0 & -3 \\ 0 & 2 & -2 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 2 & 0 & -1 & 0 & 1 \end{array} \right) \xrightarrow{r_3+2r_2} \left( \begin{array}{cccc|cccc} 1 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -5 & 1 & 0 & 0 & -3 \\ 0 & 0 & -2 & -10 & 2 & 1 & -1 & -6 \\ 0 & 0 & 0 & 2 & 0 & -1 & 0 & 1 \end{array} \right)$$

$$\xrightarrow{\begin{array}{l} r_3+5r_4 \\ \frac{5}{2}r_4 \\ r_2+\frac{5}{2}r_4 \\ r_1+r_3 \end{array}} \left( \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 2 & -4 & 0 & -1 \\ 0 & -1 & 0 & 0 & 1 & -\frac{5}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & -2 & 0 & 2 & -4 & -1 & -1 \\ 0 & 0 & 0 & 2 & 0 & -1 & 0 & 1 \end{array} \right) \xrightarrow{\begin{array}{l} -r_2 \\ -\frac{1}{2}r_3 \\ \frac{1}{2}r_4 \end{array}} \left( \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 2 & -4 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 & \frac{5}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & -1 & 2 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{array} \right)$$

$$\therefore \left( \begin{array}{cccc} 3 & 5 & 0 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 2 & 0 & 2 \end{array} \right)^{-1} = \left( \begin{array}{cccc} 2 & -4 & 0 & -1 \\ -1 & \frac{5}{2} & 0 & \frac{1}{2} \\ -1 & 2 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{array} \right)$$

1.8

1.9

1.10 1.10 设  $A = \begin{pmatrix} 4 & 2 & 3 \\ 1 & 1 & 0 \\ -1 & 2 & 3 \end{pmatrix}$ ,  $AB = A + 2B$ , 求  $B$ .

$$AB = A + 2B \Rightarrow (A - 2E)B = A \Rightarrow B = (A - 2E)^{-1}A.$$

$$(A - 2E, E) = \left( \begin{array}{ccc|ccc} 2 & 2 & 3 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ -1 & 2 & 1 & 0 & 0 & 1 \end{array} \right) \xrightarrow{\begin{array}{l} r_1-2r_2 \\ r_3+r_2 \end{array}} \left( \begin{array}{ccc|ccc} 0 & 4 & 3 & 1 & -2 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{array} \right) \rightarrow \left( \begin{array}{ccc|ccc} 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 4 & 3 & 1 & -2 & 0 \end{array} \right)$$

$$\xrightarrow{r_3-4r_2} \left( \begin{array}{ccc|ccc} 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 & -6 & -4 \end{array} \right) \xrightarrow{\begin{array}{l} r_2+r_3 \\ r_1+r_2 \\ -r_3 \end{array}} \left( \begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -4 & -3 \\ 0 & 1 & 0 & 1 & -5 & -3 \\ 0 & 0 & 1 & -1 & 6 & 4 \end{array} \right)$$

求得  $(A-2E)^{-1} = \begin{pmatrix} 1 & -4 & -3 \\ 1 & -5 & -3 \\ -1 & 6 & 4 \end{pmatrix}$ , 于是  $B = (A-2E)^{-1}A = \begin{pmatrix} 1 & -4 & -3 \\ 1 & -5 & -3 \\ -1 & 6 & 4 \end{pmatrix} \begin{pmatrix} 4 & 2 & 3 \\ 1 & 1 & 0 \\ -1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 3 & -8 & -6 \\ 2 & -9 & -6 \\ -2 & 12 & 9 \end{pmatrix}$ .

**1.11 1.11** 设  $A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ , (1) 证明  $A^k = \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix}$ ; (2) 设  $A = PAP^{-1}$ , 证明  $A^k = PA^kP^{-1}$

(1) 
$$A^2 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{bmatrix},$$

$$A^3 = A^2A = \begin{bmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} \lambda_1^3 & 0 \\ 0 & \lambda_2^3 \end{bmatrix},$$

$$\dots$$

$$A^k = A^{k-1}A = \begin{bmatrix} \lambda_1^{k-1} & 0 \\ 0 & \lambda_2^{k-1} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix}$$

(2) 
$$A = PAP^{-1}$$

$$A^2 = (PAP^{-1})(PAP^{-1}) = PA(P^{-1}P)AP^{-1} = PAEAP^{-1} = PA^2P^{-1}$$

$$A^3 = A^2A = (PA^2P^{-1})(PAP^{-1}) = PA^2(P^{-1}P)AP^{-1} = PA^2EAP^{-1} = PA^3P^{-1}$$

$$\dots$$

$$A^k = A^{k-1}A = (PA^{k-1}P^{-1})(PAP^{-1}) = PA^{k-1}(P^{-1}P)AP^{-1} = PA^{k-1}EAP^{-1} = PA^kP^{-1}$$

**1.12 1.12** 计算下列行列式

(1)  $\begin{vmatrix} 1 & 0 & 2 & 0 \\ -1 & 4 & 3 & 6 \\ 0 & 2 & -5 & 3 \\ 3 & 1 & 1 & 0 \end{vmatrix};$  (2)  $\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix};$  (3)  $\begin{vmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{vmatrix}$

(4)  $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{vmatrix};$  (5)  $\begin{vmatrix} 2 & -1 & 3 & 2 \\ 3 & -3 & 3 & 2 \\ 3 & -1 & -1 & 2 \\ 3 & -1 & 3 & -1 \end{vmatrix};$  (6)  $\begin{vmatrix} -ab & ac & ae \\ bd & -cd & de \\ bf & cf & -ef \end{vmatrix}$

(1)  $\begin{vmatrix} 1 & 0 & 2 & 0 \\ -1 & 4 & 3 & 6 \\ 0 & 2 & -5 & 3 \\ 3 & 1 & 1 & 0 \end{vmatrix} \xrightarrow[r_4-3r_1]{r_2+r_1} \begin{vmatrix} 1 & 0 & 2 & 0 \\ 0 & 4 & 5 & 6 \\ 0 & 2 & -5 & 3 \\ 0 & 1 & -5 & 0 \end{vmatrix} \xrightarrow[r_3-r_4]{r_2 \leftrightarrow r_3} \begin{vmatrix} 1 & 0 & 2 & 0 \\ 0 & 6 & 0 & 9 \\ 0 & 1 & 0 & 3 \\ 0 & 1 & -5 & 0 \end{vmatrix} \xrightarrow[r_4-3r_3]{r_2 \leftrightarrow r_3} \begin{vmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 1 & -5 & 0 \\ 0 & 2 & 0 & 3 \end{vmatrix} \xrightarrow[r_4-2r_2]{r_3-\frac{1}{3}r_4} \begin{vmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & -5 & -3 \\ 0 & 0 & 0 & -3 \end{vmatrix} = 3 \cdot 1 \cdot 1 \cdot (-5) \cdot (-3) = 45$

(2)  $\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix} \xrightarrow[r_1+r_2+r_3]{r_2 \leftrightarrow r_3} \begin{vmatrix} 2(x+y) & 2(x+y) & 2(x+y) \\ y & x+y & x \\ x+y & x & y \end{vmatrix} = 2(x+y) \begin{vmatrix} 1 & 1 & 1 \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$ 

$$\xrightarrow[r_3-(x+y)r_1]{r_2-yr_1} 2(x+y) \begin{vmatrix} 1 & 1 & 1 \\ 0 & x & x-y \\ 0 & -y & -x \end{vmatrix} = 2(x+y) \cdot 1 \cdot \begin{vmatrix} x & x-y \\ -y & -x \end{vmatrix} = 2(x+y) \cdot 1 \cdot (-x^2 + xy - y^2) = -2(x^3 - y^3).$$

(3)  $\begin{vmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{vmatrix} \xrightarrow[r_1+r_2+r_3]{r_2-r_1} \begin{vmatrix} a+2 & a+2 & a+2 \\ 1 & a & 1 \\ 1 & 1 & a \end{vmatrix} \xrightarrow[r_3-r_1]{r_2-r_1} \begin{vmatrix} a+2 & a+2 & a+2 \\ 1 & a & 1 \\ 0 & a-1 & 0 \end{vmatrix} = (a+2)(a-1)^2.$



$$(4) \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{vmatrix} \begin{matrix} r_2-r_1 \\ = \\ r_3-r_1 \\ r_4-r_1 \end{matrix} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{vmatrix} = (-2)^3 = -8.$$

$$(5) \begin{vmatrix} 2 & -1 & 3 & 2 \\ 3 & -3 & 3 & 2 \\ 3 & -1 & -1 & 2 \\ 3 & -1 & 3 & -1 \end{vmatrix} \begin{matrix} r_1-r_1 \\ = \\ i=2,3,4 \end{matrix} \begin{vmatrix} 2 & -1 & 3 & 2 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & -4 & 0 \\ 1 & 0 & 0 & -3 \end{vmatrix} \begin{matrix} c_1+\frac{1}{2}c_2+\frac{1}{4}c_3+\frac{1}{3}c_4 \\ = \end{matrix} \begin{vmatrix} \frac{35}{12} & -1 & 3 & 2 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 1 & 0 & 0 & -3 \end{vmatrix}$$

$$= \frac{35}{12} \cdot (-2) \cdot (-4) \cdot (-3) = -70.$$

$$(6) \begin{vmatrix} -ab & ac & ae \\ bd & -cd & de \\ bf & cf & -ef \end{vmatrix} = adf \begin{vmatrix} -b & c & e \\ b & -c & e \\ b & c & -e \end{vmatrix} = adf \cdot bce \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} \begin{matrix} r_1+r_2+r_3 \\ = \end{matrix} abcdef \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$\begin{matrix} r_2-r_1 \\ r_3-r_1 \end{matrix} = abcdef \begin{vmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{vmatrix} = abcdef \cdot [1 \cdot (-2) \cdot (-2)] = 4abcdef.$$

**1.13** 1.13\* 证明下列等式

$$(1) \begin{vmatrix} a^2 & 2a & 1 \\ ab & a+b & 1 \\ b^2 & 2b & 1 \end{vmatrix} = (a-b)^3 \quad (2) \begin{vmatrix} a_1+b_1x & a_1x+b_1 & c_1 \\ a_2+b_2x & a_2x+b_2 & c_2 \\ a_3+b_3x & a_3x+b_3 & c_3 \end{vmatrix} = (1-x^2) \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix};$$

$$(3) \begin{vmatrix} x & a & \cdots & a \\ a & x & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & x \end{vmatrix} = [x+(n-1)a](x-a)^{n-1}$$

$$(1) \begin{vmatrix} a^2 & 2a & 1 \\ ab & a+b & 1 \\ b^2 & 2b & 1 \end{vmatrix} \begin{matrix} r_1-2r_2+r_3 \\ = \end{matrix} \begin{vmatrix} a^2+b^2-2ab & 0 & 0 \\ ab & a+b & 1 \\ b^2 & 2b & 1 \end{vmatrix} \begin{matrix} r_1-2r_2+r_3 \\ = \end{matrix} \begin{vmatrix} (a-b)^2 & 0 & 0 \\ (a-b)b & a-b & 0 \\ b^2 & 2b & 1 \end{vmatrix} = (a-b)^3$$

$$(2) \begin{vmatrix} a_1+b_1x & a_1x+b_1 & c_1 \\ a_2+b_2x & a_2x+b_2 & c_2 \\ a_3+b_3x & a_3x+b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1+b_1x & a_1x & c_1 \\ a_2+b_2x & a_2x & c_2 \\ a_3+b_3x & a_3x & c_3 \end{vmatrix} + \begin{vmatrix} a_1+b_1x & b_1 & c_1 \\ a_2+b_2x & b_2 & c_2 \\ a_3+b_3x & b_3 & c_3 \end{vmatrix}$$

$$= \left( \begin{vmatrix} a_1 & a_1x & c_1 \\ a_2 & a_2x & c_2 \\ a_3 & a_3x & c_3 \end{vmatrix} + \begin{vmatrix} b_1x & a_1x & c_1 \\ b_2x & a_2x & c_2 \\ b_3x & a_3x & c_3 \end{vmatrix} \right) + \left( \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} b_1x & b_1 & c_1 \\ b_2x & b_2 & c_2 \\ b_3x & b_3 & c_3 \end{vmatrix} \right)$$

$$= \left( \begin{vmatrix} a_1 & a_1 & c_1 \\ x a_2 & a_2 & c_2 \\ a_3 & a_3 & c_3 \end{vmatrix} + x^2 \begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix} \right) + \left( \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + x \begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix} \right)$$

$$= \left( 0 - x^2 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \right) + \left( \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + 0 \right) = (1-x^2) \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

证法二

$$\begin{vmatrix} a_1 + b_1x & a_1x + b_1 & c_1 \\ a_2 + b_2x & a_2x + b_2 & c_2 \\ a_3 + b_3x & a_3x + b_3 & c_3 \end{vmatrix} \stackrel{c_1 - c_2}{=} (1-x) \begin{vmatrix} a_1 - b_1 & a_1x + b_1 & c_1 \\ a_2 - b_2 & a_2x + b_2 & c_2 \\ a_3 - b_3 & a_3x + b_3 & c_3 \end{vmatrix} \stackrel{c_2 + c_1}{=} (1-x^2) \begin{vmatrix} a_1 - b_1 & a_1 & c_1 \\ a_2 - b_2 & a_2 & c_2 \\ a_3 - b_3 & a_3 & c_3 \end{vmatrix}$$

$$\stackrel{c_1 - c_2}{=} (1-x^2) \begin{vmatrix} -b_1 & a_1 & c_1 \\ -b_2 & a_2 & c_2 \\ -b_3 & a_3 & c_3 \end{vmatrix} \stackrel{c_1 - c_2}{=} (1-x^2) \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$(3) \begin{vmatrix} x & a & \cdots & a \\ a & x & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & x \end{vmatrix} \stackrel{r_i + r_2 + \cdots + r_n}{=} \begin{vmatrix} x + (n-1)a & x + (n-1)a & \cdots & x + (n-1)a \\ a & x & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & x \end{vmatrix} = [x + (n-1)a] \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a & x & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & x \end{vmatrix}$$

$$\stackrel{r_i - ar_1}{=} [x + (n-1)a] \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 0 & x-a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x-a \end{vmatrix} = [x + (n-1)a](x-a)^{n-1}.$$

1.14 1.14 用克拉默法则解下列方程组:

$$(1) \begin{cases} x_1 + x_2 + x_3 + x_4 = 5 \\ x_1 + 2x_2 - x_3 + 4x_4 = -2 \\ 2x_1 - 3x_2 - x_3 - 5x_4 = -2 \\ 3x_1 + x_2 + 2x_3 + 11x_4 = 0 \end{cases} \quad (2) \begin{cases} 5x_1 + 6x_2 = 1 \\ x_1 + 5x_2 + 6x_3 = 0 \\ x_2 + 5x_3 + 6x_4 = 0 \\ x_3 + 5x_4 + 6x_5 = 0 \\ x_4 + 5x_5 = 1 \end{cases}$$

(1) 计算得  $D = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & 4 \\ 2 & -3 & -1 & -5 \\ 3 & 1 & 2 & 11 \end{vmatrix} = -142 \neq 0,$

$$D_1 = \begin{vmatrix} 5 & 1 & 1 & 1 \\ -2 & 2 & -1 & 4 \\ -2 & -3 & -1 & -5 \\ 0 & 1 & 2 & 11 \end{vmatrix} = -142, D_2 = \begin{vmatrix} 1 & 5 & 1 & 1 \\ 1 & -2 & -1 & 4 \\ 2 & -2 & -1 & -5 \\ 3 & 0 & 2 & 11 \end{vmatrix} = -284, D_3 = \begin{vmatrix} 1 & 1 & 5 & 1 \\ 1 & 2 & -2 & 4 \\ 2 & -3 & -2 & -5 \\ 3 & 1 & 0 & 11 \end{vmatrix} = -426, D_4 = \begin{vmatrix} 1 & 1 & 1 & 5 \\ 1 & 2 & -1 & -2 \\ 2 & -3 & -1 & -2 \\ 3 & 1 & 2 & 0 \end{vmatrix} = 142$$

因为系数行列式  $D \neq 0$ , 所以方程组有唯一解  $x_1 = \frac{D_1}{D} = 1, x_2 = \frac{D_2}{D} = 2, x_3 = \frac{D_3}{D} = 3, x_4 = \frac{D_4}{D} = -1.$

(2) 计算得  $D = \begin{vmatrix} 5 & 6 & 0 & 0 & 0 \\ 1 & 5 & 6 & 0 & 0 \\ 0 & 1 & 5 & 6 & 0 \\ 0 & 0 & 1 & 5 & 6 \\ 0 & 0 & 0 & 1 & 5 \end{vmatrix} = 665 \neq 0, D_1 = \begin{vmatrix} 1 & 6 & 0 & 0 & 0 \\ 0 & 5 & 6 & 0 & 0 \\ 0 & 1 & 5 & 6 & 0 \\ 0 & 0 & 1 & 5 & 6 \\ 1 & 0 & 0 & 1 & 5 \end{vmatrix} = 1507, D_2 = \begin{vmatrix} 5 & 1 & 0 & 0 & 0 \\ 1 & 0 & 6 & 0 & 0 \\ 0 & 0 & 5 & 6 & 0 \\ 0 & 0 & 1 & 5 & 6 \\ 0 & 1 & 0 & 1 & 5 \end{vmatrix} = -1145,$

$$D_3 = \begin{vmatrix} 5 & 6 & 1 & 0 & 0 \\ 1 & 5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 6 & 0 \\ 0 & 0 & 0 & 5 & 6 \\ 0 & 0 & 1 & 1 & 5 \end{vmatrix} = 703, D_4 = \begin{vmatrix} 5 & 6 & 0 & 1 & 0 \\ 1 & 5 & 6 & 0 & 0 \\ 0 & 1 & 5 & 0 & 0 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & 5 \end{vmatrix} = -395, D_5 = \begin{vmatrix} 5 & 6 & 0 & 0 & 1 \\ 1 & 5 & 6 & 0 & 0 \\ 0 & 1 & 5 & 6 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix} = 212$$

因为系数行列式  $D \neq 0$ , 所以方程组有唯一解

$$x_1 = \frac{D_1}{D} = \frac{1507}{665}, x_2 = \frac{D_2}{D} = -\frac{1145}{665}, x_3 = \frac{D_3}{D} = \frac{703}{665}, x_4 = \frac{D_4}{D} = -\frac{395}{665}, x_5 = \frac{D_5}{D} = -\frac{212}{665}.$$

1.15 1.15 求下列方阵的逆阵

$$(1) \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}; \quad (2) \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}; \quad (3) \begin{pmatrix} 1 & 2 & -1 \\ 3 & 4 & -2 \\ 5 & -4 & 1 \end{pmatrix}; \quad (4) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 2 & 1 & 3 & 0 \\ 1 & 2 & 1 & 4 \end{pmatrix};$$

(1) 套用公式  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} (ad-bc \neq 0)$ , 得  $\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = \frac{1}{1 \cdot 5 - 2 \cdot 2} \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$ .

(2) 套用上述公式, 得  $\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} = \frac{1}{\cos^2 \theta + \sin^2 \theta} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$ .

(3)  $\left[ \begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 3 & 4 & -2 & 0 & 1 & 0 \\ 5 & -4 & 1 & 0 & 0 & 1 \end{array} \right] \xrightarrow[r_3-5r_1]{r_2-3r_1} \left[ \begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -2 & 1 & -3 & 1 & 0 \\ 0 & -14 & 6 & -5 & 0 & 1 \end{array} \right] \xrightarrow[r_3-7r_2]{r_1+r_2} \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 1 & 0 \\ 0 & -2 & 1 & -3 & 1 & 0 \\ 0 & 0 & -1 & 16 & -7 & 1 \end{array} \right]$

$\xrightarrow{r_2+r_3} \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 1 & 0 \\ 0 & -2 & 0 & 13 & -6 & 1 \\ 0 & 0 & -1 & 16 & -7 & 1 \end{array} \right] \xrightarrow[-r_3]{-\frac{1}{2}r_2} \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 1 & 0 \\ 0 & 1 & 0 & -\frac{13}{2} & 3 & -\frac{1}{2} \\ 0 & 0 & -1 & -16 & 7 & -1 \end{array} \right], \text{ 得 } \begin{bmatrix} 1 & 2 & -1 \\ 3 & 4 & -2 \\ 5 & -4 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 1 & 0 \\ -\frac{13}{2} & 3 & -\frac{1}{2} \\ -16 & 7 & -1 \end{bmatrix}$ .

(4)  $\left[ \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 0 & 0 & 1 & 0 \\ 1 & 2 & 1 & 4 & 0 & 0 & 0 & 1 \end{array} \right] \xrightarrow[r_3-2r_1]{r_4-r_1, r_2-r_1} \left[ \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 4 & 0 & -1 & 0 & 1 \end{array} \right]$

$\xrightarrow[r_4-\frac{1}{3}r_3]{r_3-\frac{1}{2}r_2} \left( \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & -\frac{3}{2} & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 4 & \frac{1}{2} & -\frac{5}{6} & -\frac{1}{3} & 1 \end{array} \right) \rightarrow \left( \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{6} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{8} & -\frac{5}{24} & -\frac{1}{12} & \frac{1}{4} \end{array} \right)$

得  $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 2 & 1 & 3 & 0 \\ 1 & 2 & 1 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{6} & \frac{1}{3} & 0 \\ \frac{1}{8} & -\frac{5}{24} & -\frac{1}{12} & \frac{1}{4} \end{pmatrix} = \frac{1}{24} \begin{pmatrix} 24 & 0 & 0 & 0 \\ -12 & 12 & 0 & 0 \\ -12 & -4 & 8 & 0 \\ 3 & -5 & -2 & 6 \end{pmatrix}$ .

**1.16** 1.16. 解下列矩阵方程

(1)  $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \mathbf{X} = \begin{bmatrix} 4 & -6 \\ 2 & 1 \end{bmatrix}$  (2)  $\mathbf{X} \begin{bmatrix} 2 & 1 & -1 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 3 \\ 4 & 3 & 2 \end{bmatrix}$  (3)  $\begin{bmatrix} 1 & 4 \\ -1 & 2 \end{bmatrix} \mathbf{X} \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 0 & -1 \end{bmatrix}$

(1)  $\mathbf{X} = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 4 & -6 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 4 & -6 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -23 \\ 0 & 8 \end{bmatrix}$

(2)  $\mathbf{X} = \begin{bmatrix} 1 & -1 & 3 \\ 4 & 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 & 3 \\ 4 & 3 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 & \frac{1}{3} \\ -\frac{2}{3} & 1 & -\frac{2}{3} \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 2 & 1 \\ -\frac{8}{3} & 5 & -\frac{2}{3} \end{bmatrix}$

$$(3) X = \begin{bmatrix} 1 & 4 \\ -1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}^{-1} = \frac{1}{6} \begin{bmatrix} 2 & -4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & -1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1/4 & 0 \end{bmatrix}$$

**1.17 1.17** 设  $A$  是  $n$  阶矩阵,  $A^*$  为其伴随矩阵, 证明:

(1) 若  $|A|=0$ , 则  $|A^*|=0$

(2)  $|A^*|=|A|^{n-1}$ .

(1) 设  $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$ , 则  $A^* = \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix}$ . 如果  $A$  的第一行元素全为零, 则  $A_{ij}=0 (i=2, \dots, n, j=1, \dots, n)$ ,

于是  $|A^*|=0$ . 假设  $A$  的第一行元素不全为零, 例如  $a_{11} \neq 0$ , 作如下行初等变换, 得

$$|A^*| = \begin{vmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{vmatrix} \stackrel{a_{11}r_1 + a_{12}r_2 + \cdots + a_{1n}r_n}{=} \frac{1}{a_{11}} \begin{vmatrix} |A| & 0 & \cdots & 0 \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{vmatrix}.$$

现  $|A|=0$ , 因此  $|A^*|=0$ .

(2) 一般地,  $|A^*A| = ||A|E| = |A|^n|E| = |A|^n$ , 但  $|A^*A| = |A^*||A|$ . 于是  $|A^*||A| = |A|^n$ . 从而, 若  $|A| \neq 0$ , 立刻得到  $|A^*| = |A|^{n-1}$ . 而若

$|A|=0$ , 由(1)知  $|A^*| = |A|^{n-1}$  仍成立.

**1.18 1.18** 设  $A = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}$ ,  $B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 4 & 2 \end{bmatrix}$

利用分块矩阵的乘法, 计算  $AB$ .

$$AB = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 4 & 2 \end{bmatrix} = \begin{bmatrix} A_1 & O \\ O & -2E \end{bmatrix} \begin{bmatrix} B_1 & O \\ O & B_2 \end{bmatrix} = \begin{bmatrix} A_1B_1 & O \\ O & -2B_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 3 & 0 & 0 \\ 0 & 4 & 3 & 0 & 0 \\ 3 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -6 \\ 0 & 0 & 0 & -8 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 & 0 & 0 \\ 0 & 4 & 3 & 0 & 0 \\ 3 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -6 \\ 0 & 0 & 0 & -8 & -4 \end{bmatrix}$$

**1.19 1.19** 若  $AB=BA, AC=CA$ , 证明:  $A(B+C) = (B+C)A$ .

$$A(B+C) = AB+AC = BA+CA = (B+C)A.$$

**1.20 1.20** 设  $A, B$  为  $n$  阶方阵, 证明:  $|AB| = |BA|$ .

$$|AB| = |A||B| = |B||A| = |BA|.$$

**1.21** 1.21 设3阶方阵  $A$  的伴随矩阵为  $A^*$  且  $|A| = \frac{1}{2}$ , 求  $|(3A)^{-1} - 2A^*|$ .

$$\begin{aligned} |(3A)^{-1} - 2A^*| &= |3^{-1}A^{-1} - 2A^*| = |(3^{-1}E - 2A^*A)A^{-1}| = |(3^{-1}E - 2|A|E)A^{-1}| \\ &= |(3^{-1} - 2|A|)EA^{-1}| = \left| \left( 3^{-1} - 2 \cdot \frac{1}{2} \right) \cdot A^{-1} \right| = \left| \left( -\frac{2}{3} \right) A^{-1} \right| = \left( -\frac{2}{3} \right)^3 |A^{-1}| = \left( -\frac{2}{3} \right)^3 \frac{1}{\frac{1}{2}} = -\frac{16}{27} \end{aligned}$$

或  $|(3A)^{-1} - 2A^*| = \left| \frac{1}{3|A|} A^* - 2A^* \right| = \left| \left( \frac{2}{3} - 2 \right) A^* \right| = \left( -\frac{4}{3} \right)^3 |A|^2 = \left( -\frac{4}{3} \right)^3 \left( \frac{1}{2} \right)^2 = -\frac{16}{27}$ .

**1.22** 1.22\* 设  $A$  为  $n$  阶方阵,  $A^k = 0, k \in \mathbb{N}$ , 求证  $E - A$  可逆, 并写出逆矩阵的表达式.

$$\begin{aligned} &\because (E - A)(E + A + A^2 + \dots + A^{k-1}) \\ &= (E + A + A^2 + \dots + A^{k-1}) - (A + A^2 + A^3 + \dots + A^k) \\ &= E - A^k = E - 0 = E, \end{aligned}$$

$$\therefore E - A \text{ 可逆, 且 } (E - A)^{-1} = E + A + A^2 + \dots + A^{k-1}.$$

**1.23** 1.23 设分块阵  $X = \begin{bmatrix} O & A \\ B & O \end{bmatrix}$ , 其中  $A, B$  可逆, 求  $X^{-1}$ .

$$X^{-1} = \begin{bmatrix} O & A^{-1} \\ B^{-1} & O \end{bmatrix}. \text{ 验算 } \begin{bmatrix} O & A \\ B & O \end{bmatrix} \begin{bmatrix} O & B^{-1} \\ A^{-1} & O \end{bmatrix} = \begin{bmatrix} AA^{-1} & O \\ O & BB^{-1} \end{bmatrix} = \begin{bmatrix} E & O \\ O & E \end{bmatrix} = E. \text{ OK}$$

**1.24** 1.24 设  $A = \begin{bmatrix} 0 & a_1 & 0 & 0 & \dots & 0 \\ 0 & 0 & a_2 & 0 & \dots & 0 \\ 0 & 0 & 0 & a_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & a_{n-1} \\ a_n & 0 & 0 & 0 & \dots & 0 \end{bmatrix}$  ( $a_i \neq 0, i = 1, 2, \dots, n$ ), 求  $A^{-1}$ .

$$\begin{aligned} A - E &= \left( \begin{array}{cccccc|cccc} 0 & a_1 & 0 & 0 & \dots & 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & a_2 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & a_3 & \dots & 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & a_{n-1} & 0 & 0 & 0 & \dots & 1 & 0 \\ a_n & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{array} \right) \rightarrow \left( \begin{array}{cccccc|cccc} a_n & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & a_1 & 0 & 0 & \dots & 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & a_2 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & a_3 & \dots & 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & a_{n-1} & 0 & 0 & 0 & \dots & 1 & 0 \end{array} \right) \\ &\rightarrow \left( \begin{array}{cccccc|cccc} 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 1/a_n \\ 0 & 1 & 0 & 0 & \dots & 0 & 1/a_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 & 1/a_2 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 & 0 & 1/a_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 1/a_{n-1} & 0 \end{array} \right); A^{-1} = \left( \begin{array}{cccccc|cccc} 0 & 0 & 0 & \dots & 0 & 1/a_n \\ 1/a_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1/a_2 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1/a_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1/a_{n-1} & 0 \end{array} \right) \end{aligned}$$

**1.25** 1.25 设  $A, B$  均为  $n$  阶方阵且  $|A| = 2, |B| = -3$ , 求  $|2(A^*)B^{-1}|$ .

$$|2A^*B^{-1}| = 2^n |A^*| |B^{-1}| = 2^n |A|^{n-1} |B|^{-1} = 2^n \cdot 2^{n-1} \cdot \frac{1}{-3} = -\frac{2^{2n-1}}{3}$$

注:  $AA^* = |A|E \Rightarrow |A| |A^*| = |A|^n |E| \Rightarrow |A^*| = |A|^{n-1}$ .

**1.26** 1.26 设  $A$  为  $n$  阶非奇异(可逆)矩阵, 其伴随阵为  $A^*$ , 求  $(A^*)^*$

$$(A^*)^* = |A^*| (A^*)^{-1} = |A^*| \frac{1}{|A|} A = |A|^{n-1} \frac{1}{|A|} A = |A|^{n-2} A$$

$$\text{或 } (A^*)^* = (|A| A^{-1})^* = |A| A^{-1} (|A| A^{-1})^{-1} = (|A|^n |A|^{-1}) (|A|^{-1} A) = \dots$$

**1.27** 1.27.

**1.28** 1.28..

## 习题二

2.1 2.1 讨论下列向量组的线性相关性

(1)  $\alpha_1 = (2, -1, 0), \alpha_2 = (-1, 1, 3), \alpha_3 = (1, 0, 3)$

(2)  $\alpha_1 = (-1, 3, 4), \alpha_2 = (2, 0, 1)$

(3)  $\alpha_1 = (1, 2, -1, 2), \alpha_2 = (3, -1, 0, 1), \alpha_3 = (2, -1, 3, 2), \alpha_4 = (1, 0, -3, -1)$

(4)  $\alpha_1 = (2, 1, 0), \alpha_2 = (-1, 3, 2), \alpha_3 = (0, 3, 4), \alpha_4 = (-1, 5, 6)$

$$(1) (\alpha_1^T, \alpha_2^T, \alpha_3^T) = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 1 & 0 \\ 0 & 3 & 3 \end{pmatrix} \xrightarrow[r_3-3r_1]{r_1+2r_2} \begin{pmatrix} 0 & 1 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow[r_3-3r_1]{r_1 \leftrightarrow r_2} \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

可见  $R\{\alpha_1, \alpha_2, \alpha_3\} = 2 < m = 3$ , 故向量组线性相关.

$$(2) (\alpha_1^T, \alpha_2^T) = \begin{pmatrix} -1 & 2 \\ 3 & 0 \\ 4 & 1 \end{pmatrix} \xrightarrow[r_3-4r_2]{\frac{1}{3}r_2} \begin{pmatrix} 0 & 2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \xrightarrow{r_1-r_3} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

可见  $R\{\alpha_1, \alpha_2\} = 2 = m = 2$ , 故向量组线性无关.

$$(3) (\alpha_1^T, \alpha_2^T, \alpha_3^T, \alpha_4^T) = \begin{pmatrix} 1 & 3 & 2 & 1 \\ 2 & -1 & -1 & 0 \\ -1 & 0 & 3 & -3 \\ 2 & 1 & 2 & -1 \end{pmatrix} \xrightarrow[r_3+2r_1]{r_1+r_3} \begin{pmatrix} 0 & 3 & 5 & -2 \\ 0 & -1 & 5 & -6 \\ -1 & 0 & 3 & -3 \\ 0 & 2 & 3 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 0 & 3 & -3 \\ 0 & -1 & 5 & -6 \\ 0 & 3 & 5 & -2 \\ 0 & 2 & 3 & -1 \end{pmatrix}$$

$$\xrightarrow[r_4+2r_2]{r_3+3r_2} \begin{pmatrix} -1 & 0 & 3 & -3 \\ 0 & -1 & 5 & -6 \\ 0 & 0 & 20 & -20 \\ 0 & 0 & 13 & -13 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 0 & 3 & -3 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

可见  $R\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} = 3 < m = 4$ , 故向量组线性相关.

$$(4) (\alpha_1^T, \alpha_2^T, \alpha_3^T, \alpha_4^T) = \begin{pmatrix} 2 & -1 & 0 & -1 \\ 1 & 3 & 3 & 5 \\ 0 & 2 & 4 & 6 \end{pmatrix} \xrightarrow[r_3-2r_2]{r_2-\frac{1}{2}r_1} \begin{pmatrix} 2 & -1 & 0 & -1 \\ 0 & 7 & 3 & 11 \\ 0 & 2 & 4 & 6 \end{pmatrix} \xrightarrow[r_3-2r_2]{\frac{2}{7}r_2} \begin{pmatrix} 2 & -1 & 0 & -1 \\ 0 & 1 & \frac{6}{7} & \frac{11}{7} \\ 0 & 0 & \frac{16}{7} & \frac{20}{7} \end{pmatrix} \xrightarrow[r_1+r_2]{\frac{7}{16}r_2} \begin{pmatrix} 2 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{5}{4} \end{pmatrix}$$

可见  $R\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} = 3 < m = 4$ , 故向量组线性相关.

解法二 现有 4 个 3 维,  $4 > 3$ , 所以给出的向量组线性相关.

**P45 推论 2.1** 任意  $m(m > n)$  个  $n$  维向量线性相关.

2.2 2.2 求下列矩阵的秩

(1)  $\begin{pmatrix} 1 & 2 & -1 & 0 \\ 3 & 2 & -1 & 2 \\ 1 & -1 & 3 & 3 \end{pmatrix}$

(2)  $\begin{pmatrix} 2 & -1 & 3 & 0 \\ 1 & 2 & 1 & 1 \\ 2 & -1 & 1 & 3 \\ 2 & 1 & 4 & 0 \end{pmatrix}$

(3)  $\begin{pmatrix} 2 & -1 & 0 & 3 \\ 1 & 2 & -1 & 2 \\ 3 & 1 & -1 & 5 \end{pmatrix}$

$$(1) \begin{pmatrix} 1 & 2 & -1 & 0 \\ 3 & 2 & -1 & 2 \\ 1 & -1 & 3 & 3 \end{pmatrix} \xrightarrow{\substack{r_3-3r_1 \\ r_3-r_1}} \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & -4 & 2 & 2 \\ 0 & -3 & 4 & 3 \end{pmatrix} \xrightarrow{\substack{-\frac{1}{4}r_2 \\ r_3+3r_2 \\ r_1-2r_2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{5}{2} & \frac{3}{2} \end{pmatrix} \xrightarrow{\substack{\frac{2}{5}r_3 \\ r_2+\frac{1}{2}r_3}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -\frac{1}{5} \\ 0 & 0 & 1 & \frac{3}{5} \end{pmatrix}$$

可见秩  $R=3$ .

$$(2) \begin{pmatrix} 2 & -1 & 3 & 0 \\ 1 & 2 & 1 & 1 \\ 2 & -1 & 1 & 3 \\ 2 & 1 & 4 & 0 \end{pmatrix} \xrightarrow{\substack{r_2-\frac{1}{2}r_1 \\ r_3-r_1 \\ r_4-r_1}} \begin{pmatrix} 2 & -1 & 3 & 0 \\ 0 & \frac{5}{2} & -\frac{1}{2} & 1 \\ 0 & 0 & -2 & 3 \\ 0 & 2 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & -1 & 3 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 5 & -1 & 1 \\ 0 & 0 & -2 & 3 \end{pmatrix} \xrightarrow{\substack{r_3-\frac{5}{2}r_2 \\ r_4+\frac{4}{7}r_3}} \begin{pmatrix} 2 & -1 & 3 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & -\frac{7}{2} & 2 \\ 0 & 0 & 0 & \frac{29}{7} \end{pmatrix}$$

可见秩  $R=4$ .

$$\begin{pmatrix} 2 & -1 & 3 & 0 \\ 1 & 2 & 1 & 1 \\ 2 & -1 & 1 & 3 \\ 2 & 1 & 4 & 0 \end{pmatrix} \xrightarrow{\substack{r_3-r_1 \\ r_4-r_1 \\ r_1-2r_2}} \begin{pmatrix} 0 & -5 & 1 & -2 \\ 1 & 2 & 1 & 1 \\ 0 & 0 & -2 & 3 \\ 0 & 2 & 1 & 0 \end{pmatrix} \xrightarrow{r_1+\frac{5}{2}r_2} \begin{pmatrix} 0 & 0 & \frac{7}{2} & -2 \\ 1 & 2 & 1 & 1 \\ 0 & 0 & -2 & 3 \\ 0 & 2 & 1 & 0 \end{pmatrix} \xrightarrow{r_1+\frac{7}{4}r_3} \begin{pmatrix} 0 & 0 & 0 & \frac{13}{4} \\ 1 & 2 & 1 & 1 \\ 0 & 0 & -2 & 3 \\ 0 & 2 & 1 & 0 \end{pmatrix}$$

或

$$\rightarrow \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & 0 & \frac{13}{4} \end{pmatrix}, R=4$$

$$(3) \begin{pmatrix} 2 & -1 & 0 & 3 \\ 1 & 2 & -1 & 2 \\ 3 & 1 & -1 & 5 \end{pmatrix} \xrightarrow{\substack{r_3-3r_2 \\ r_2-\frac{1}{2}r_1}} \begin{pmatrix} 2 & -1 & 0 & 3 \\ 0 & \frac{5}{2} & -1 & \frac{1}{2} \\ 0 & -5 & 2 & -1 \end{pmatrix} \xrightarrow{\substack{2r_2 \\ r_3+r_2}} \begin{pmatrix} 2 & -1 & 0 & 3 \\ 0 & 5 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, R=2$$

**2.3** 2.3 求解下列齐次线性方程组

$$(1) \begin{cases} x_1 + 2x_2 - x_3 + 2x_4 = 0 \\ x_1 - x_2 + 2x_3 - x_4 = 0 \\ 2x_1 + x_2 + x_3 + x_4 = 0 \end{cases}; \quad (2) \begin{cases} 2x_1 + x_2 + x_3 = 0 \\ 3x_1 - 3x_2 + x_3 = 0 \\ x_1 + 2x_2 - 3x_3 = 0 \end{cases}; \quad (3) \begin{cases} x_1 - 2x_2 - 3x_3 + 4x_4 = 0 \\ 3x_1 - 6x_2 - 9x_3 + 12x_4 = 0 \\ 2x_1 - 4x_2 - 6x_3 + 8x_4 = 0 \end{cases}$$

(1) 对方程组的系数矩阵作行初等变换

$$\begin{pmatrix} 1 & 2 & -1 & 2 \\ 1 & -1 & 2 & -1 \\ 2 & 1 & 1 & 1 \end{pmatrix} \xrightarrow{\substack{r_2-r_1 \\ r_3-2r_1}} \begin{pmatrix} 1 & 2 & -1 & 2 \\ 0 & -3 & 3 & -3 \\ 0 & -3 & 3 & -3 \end{pmatrix} \xrightarrow{\substack{r_3-r_2 \\ -\frac{1}{3}r_2 \\ r_1-2r_2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

得简化行阶梯形(Reduced row echelon form, RREF). 对应的同解方程组为  $\begin{cases} x_1 + x_3 = 0 \\ x_2 - x_3 + x_4 = 0 \end{cases}$ ,

$$\text{方程组的解为 } \mathbf{x} = \begin{pmatrix} -k_1 \\ k_1 - k_2 \\ k_1 \\ k_2 \end{pmatrix} = k_1 \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \quad (k_1, k_2 \in \mathbb{R}).$$

(2) 对方程组的系数矩阵作行初等变换



$$\begin{pmatrix} 2 & 1 & 1 \\ 3 & -3 & 1 \\ 1 & 2 & -3 \end{pmatrix} \xrightarrow[r_2 - \frac{3}{2}r_1]{r_2 - \frac{3}{2}r_1} \begin{pmatrix} 2 & 1 & 1 \\ 0 & -\frac{9}{2} & -\frac{1}{2} \\ 0 & \frac{3}{2} & -\frac{7}{2} \end{pmatrix} \xrightarrow[2r_3]{2r_2} \begin{pmatrix} 2 & 1 & 1 \\ 0 & -9 & -1 \\ 0 & 3 & -7 \end{pmatrix} \xrightarrow[r_3 + \frac{1}{3}r_2]{r_3 + \frac{1}{3}r_2} \begin{pmatrix} 2 & 1 & 1 \\ 0 & -9 & -1 \\ 0 & 0 & -\frac{22}{3} \end{pmatrix}, R=3=n, \text{ 方程组有唯一零解 } \mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

(3) 对方程组的系数矩阵作行初等变换

$$A = \begin{pmatrix} 1 & -2 & -3 & 4 \\ 3 & -6 & -9 & 12 \\ 2 & -4 & -6 & 8 \end{pmatrix} \xrightarrow[r_3 - 2r_1]{r_2 - 3r_1} \begin{pmatrix} 1 & -2 & -3 & 4 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix}, \text{ 得 } \mathbf{x} = \begin{pmatrix} 2k_1 + 3k_2 - 4k_3 \\ k_1 \\ k_2 \\ k_3 \end{pmatrix} = k_1 \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + k_3 \begin{pmatrix} -4 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad (k_1, k_2, k_3 \in \mathbb{R})$$

**2.4** 2.4 求一个齐次线性方程组使他的基础解系为

$$\alpha_1 = (1, -1, 2, 0)^T, \alpha_2 = (0, 2, -2, 3)^T$$

由题意, 齐次线性方程组的通解为

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = k_1 \begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} 0 \\ 2 \\ -2 \\ 3 \end{pmatrix} \quad (k_1, k_2 \in \mathbb{R}), \text{ 或 } \begin{cases} x_1 = k_1 \\ x_2 = -k_1 + 2k_2 \\ x_3 = 2k_1 - 2k_2 \\ x_4 = 3k_2 \end{cases}.$$

从中消去  $k_1, k_2$ , 得 
$$\begin{cases} x_1 + x_2 - \frac{2}{3}x_4 = 0 \\ 2x_1 - x_3 - \frac{2}{3}x_4 = 0 \end{cases} \text{ 即为所求.}$$

解法二: 设所求的齐次线性方程组为

$$\begin{cases} x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0 \\ x_2 + b_3x_3 + b_4x_4 = 0 \end{cases}$$

将  $\alpha_1 = (1, -1, 2, 0)^T, \alpha_2 = (0, 2, -2, 3)^T$  分别代入方程组, 得

$$\begin{cases} 1 - a_2 + 2a_3 = 0 \\ 2a_2 - 2a_3 + 3a_4 = 0 \end{cases} \dots\dots(1), \quad \begin{cases} -1 + 2b_3 = 0 \\ 2 - 2b_3 + 3b_4 = 0 \end{cases} \dots\dots(2)$$

解方程组(1), 得其中一个解  $a_2 = -1, a_3 = -1, a_4 = 0$ . 解方程组(2), 得其中一个解  $b_3 = 1/2, b_4 = -1/3$ . 从而得一个满足要求的方

程组 
$$\begin{cases} x_1 - x_2 - x_3 = 0 \\ x_2 + \frac{1}{2}x_3 - \frac{1}{3}x_4 = 0 \end{cases}$$

**2.5** 2.5 求下列非齐次线性方程组的通解

$$(1) \begin{cases} 2x_1 - x_2 + 2x_3 - x_4 = 1 \\ -x_1 + 2x_2 - x_3 + 2x_4 = 2 \\ x_1 + x_2 + x_3 + x_4 = 3 \end{cases} \quad (2) \begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 3x_2 + x_3 = -1 \\ 3x_1 + x_2 + 2x_3 = 2 \end{cases} \quad (3) \begin{cases} 2x_1 + 3x_2 - x_3 + x_4 = 2 \\ 2x_1 - 2x_2 + 3x_3 - 3x_4 = -1 \\ 4x_1 + x_2 + 2x_3 - 2x_4 = 2 \end{cases}$$

(1) 对方程组的增广矩阵作行初等变换, 将之化为简化行阶梯形

$$\left( \begin{array}{cccc|c} 2 & -1 & 2 & -1 & 1 \\ -1 & 2 & -1 & 2 & 2 \\ 1 & 1 & 1 & 1 & 3 \end{array} \right) \xrightarrow[r_1 - 2r_3]{r_2 + r_3} \left( \begin{array}{cccc|c} 0 & -3 & 0 & -3 & -5 \\ 0 & 3 & 0 & 3 & 5 \\ 1 & 1 & 1 & 1 & 3 \end{array} \right) \xrightarrow[r_3 + r_2]{r_3 \leftrightarrow r_1} \left( \begin{array}{cccc|c} 1 & 1 & 1 & 1 & 3 \\ 0 & 3 & 0 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right) \xrightarrow[r_1 - r_2]{\frac{1}{3}r_2} \left( \begin{array}{cccc|c} 1 & 0 & 1 & 0 & \frac{4}{3} \\ 0 & 1 & 0 & 1 & \frac{5}{3} \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

$$\text{立刻得到方程组的解 } \mathbf{x} = k_1 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{4}{3} \\ \frac{5}{3} \\ 0 \\ 0 \end{pmatrix} \quad (k_1, k_2 \in \mathbb{R})$$

(2) 对方程组的增广矩阵作行初等变换, 将之化为简化行阶梯形

$$\left( \begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 3 & 1 & -1 \\ 3 & 1 & 2 & 2 \end{array} \right) \xrightarrow[r_3-3r_1]{r_2-2r_1} \left( \begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -5 & -3 \\ 0 & -5 & -7 & -1 \end{array} \right) \xrightarrow[r_3+5r_2]{r_1-2r_2} \left( \begin{array}{ccc|c} 1 & 0 & -7 & -5 \\ 0 & 1 & 5 & 3 \\ 0 & 0 & 18 & 14 \end{array} \right) \xrightarrow[r_1+7r_3]{\frac{1}{18}r_3} \left( \begin{array}{ccc|c} 1 & 0 & 0 & \frac{4}{9} \\ 0 & 1 & 0 & -\frac{8}{9} \\ 0 & 0 & 1 & \frac{7}{9} \end{array} \right), \mathbf{x} = \begin{pmatrix} \frac{4}{9} \\ -\frac{8}{9} \\ \frac{7}{9} \\ 0 \end{pmatrix}$$

$$\text{立刻得到方程组的解 } \mathbf{x} = k_1 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{4}{3} \\ \frac{5}{3} \\ 0 \\ 0 \end{pmatrix} \quad (k_1, k_2 \in \mathbb{R})$$

(3) 对方程组的增广矩阵作行初等变换

$$\left( \begin{array}{cccc|c} 2 & 3 & -1 & 1 & 2 \\ 2 & -2 & 3 & -3 & -1 \\ 4 & 1 & 2 & -2 & 2 \end{array} \right) \xrightarrow[r_3-2r_1]{r_2-r_1} \left( \begin{array}{cccc|c} 2 & 3 & -1 & 1 & 2 \\ 0 & -5 & 4 & -4 & -3 \\ 0 & -5 & 4 & -4 & -2 \end{array} \right) \xrightarrow{r_3-r_2} \left( \begin{array}{cccc|c} 2 & 3 & -1 & 1 & 2 \\ 0 & -5 & 4 & -4 & -3 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right)$$

因为  $R(\bar{A}) \neq R(A)$ , 所以方程组无解.

**2.6** 2.6 若向量组  $\alpha_1, \alpha_2, \alpha_3$  线性无关,  $\alpha_1, \alpha_2, \alpha_4$  线性相关. 试证  $\alpha_4$  可由  $\alpha_1, \alpha_2, \alpha_3$  线性表示.

$\alpha_1, \alpha_2, \alpha_3$  线性无关  $\Rightarrow \alpha_1, \alpha_2$  线性无关.

$\alpha_1, \alpha_2, \alpha_4$  线性相关  $\Rightarrow \alpha_4$  可由  $\alpha_1, \alpha_2$  线性表示.

从而  $\alpha_4$  可由  $\alpha_1, \alpha_2, \alpha_3$  线性表示.

**证法二**

$\alpha_1, \alpha_2, \alpha_4$  线性相关  $\Rightarrow \alpha_1, \alpha_2, \alpha_3, \alpha_4$  线性相关.

$\alpha_1, \alpha_2, \alpha_3$  线性无关  $\Rightarrow \alpha_4$  可由  $\alpha_1, \alpha_2, \alpha_3$  线性表示.

注意: “ $\because \alpha_1, \alpha_2, \alpha_3$  线性无关,  $\therefore$  存在全为 0 的  $k_1, k_2, k_3$ , 使得  $k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 = 0$ .” 这个说法是有问题的, 因为不管是否相关, 这些  $k_1, k_2, k_3$  总是存在的!

**2.7** 2.7 设线性方程组

$$\begin{cases} (1+\lambda)x_1 + x_2 + x_3 = 0 \\ x_1 + (1+\lambda)x_2 + x_3 = \lambda \\ x_1 + x_2 + (1+\lambda)x_3 = -\lambda^2 \end{cases}$$

当  $\lambda$  等于何值时, (1) 方程组有唯一解; (2) 无解; (3) 有无穷多解. 并求此时方程组的通解.

$$|A| = \begin{vmatrix} 1+\lambda & 1 & 1 \\ 1 & 1+\lambda & 1 \\ 1 & 1 & 1+\lambda \end{vmatrix} = \begin{vmatrix} 3+\lambda & 1 & 1 \\ 3+\lambda & 1+\lambda & 1 \\ 3+\lambda & 1 & 1+\lambda \end{vmatrix} = \begin{vmatrix} 3+\lambda & 1 & 1 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{vmatrix} = \lambda^2(3+\lambda).$$

(1)  $\lambda \neq 0, -3$  时方程组有唯一解.

(2)  $\lambda = -3$  时,  $\bar{A} = \begin{pmatrix} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & -3 \\ 1 & 1 & -2 & -9 \end{pmatrix} \xrightarrow[r_3-r_2]{r_1+2r_2} \begin{pmatrix} 0 & -3 & 3 & -6 \\ 1 & -2 & 1 & -3 \\ 0 & 3 & -3 & -6 \end{pmatrix} \xrightarrow{r_3+r_1} \begin{pmatrix} 0 & -3 & 3 & -6 \\ 1 & -2 & 1 & -3 \\ 0 & 0 & 0 & -12 \end{pmatrix}$ . 无解

$$(2) \lambda = 0 \text{ 时, } \bar{A} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, R(\bar{A}) = R(A) = 1 < 3, \text{ 有无穷多解.}$$

$$\bar{A} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, R(\bar{A}) = R(A) = 1 < 3$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -k_1 - k_2 \\ k_1 \\ k_2 \end{pmatrix} = k_1 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad (k_1, k_2 \in \mathbb{R})$$

**2.8** 2.8 设  $\alpha_1 = (x, -1, 1)$ ,  $\alpha_2 = (0, 3, 3)$ ,  $\alpha_3 = (1, 2, 0)$

(1) 当  $x$  为何值时, 向量组  $\alpha_1, \alpha_2, \alpha_3$  线性相关.

(2) 当  $x$  为何值时, 向量组  $\alpha_1, \alpha_2, \alpha_3$  线性无关.

(3) 当  $\alpha_1, \alpha_2, \alpha_3$  线性相关时, 将  $\alpha_1$  表示为  $\alpha_2, \alpha_3$  的线性组合.

$$|\alpha_1' \alpha_2' \alpha_3'| = \begin{vmatrix} x & 0 & 1 \\ -1 & 3 & 2 \\ 1 & 3 & 0 \end{vmatrix} = \begin{vmatrix} x & 0 & 1 \\ -1-2x & 3 & 0 \\ 1 & 3 & 0 \end{vmatrix} = -6(1+x)$$

(1) 当  $x = -1$  时,  $\alpha_1, \alpha_2, \alpha_3$  线性相关;

(2) 当  $x \neq -1$  时,  $\alpha_1, \alpha_2, \alpha_3$  线性无关.

(3) 现  $x = -1$ . 设  $\alpha_1$  可表示为  $\alpha_2, \alpha_3$  的线性组合:  $\alpha_1 = x_1 \alpha_2 + x_2 \alpha_3$ , 即  $\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = x_1 \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ . 则有线性方程组

$$\begin{pmatrix} \alpha_1^T & \alpha_2^T \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \text{ 或 } \begin{pmatrix} 0 & 1 \\ 3 & 2 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.$$

$$\bar{A} = \left( \begin{array}{cc|c} 0 & 1 & -1 \\ 3 & 2 & -1 \\ 3 & 0 & 1 \end{array} \right) \rightarrow \left( \begin{array}{cc|c} 3 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array} \right), \text{ 得 } x_1 = \frac{1}{3}, x_2 = -1. \text{ 于是 } \alpha_1 = \frac{1}{3} \alpha_2 - x_2 \alpha_3$$

**2.9** 2.9.

**2.10** 2.10. 设线性方程组 
$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = 0 \\ \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = 0 \end{cases}$$
 的解都是  $b_1x_1 + b_2x_2 + \dots + b_nx_n = 0$  的解. 试证  $\beta = (b_1, b_2, \dots, b_n)$  是

$$\begin{aligned} \alpha_1 &= (a_{11} \ a_{12} \ \dots \ a_{1n}) \\ \alpha_2 &= (a_{21} \ a_{22} \ \dots \ a_{2n}) \\ &\dots \ \dots \\ \alpha_m &= (a_{m1} \ a_{m2} \ \dots \ a_{mn}) \end{aligned}$$

的线性组合.

$$\text{方程组} \begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = 0 \\ \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = 0 \end{cases} \text{与} \begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = 0 \\ \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = 0 \\ b_1x_1 + b_2x_2 + \dots + b_nx_n = 0 \end{cases}$$

是同解方程组，它们的基础解系相同，从而它们的系数

矩阵的秩相同，即向量组  $\alpha_1, \alpha_2, \dots, \alpha_m$  和向量组  $\alpha_1, \alpha_2, \dots, \alpha_m, \beta$  有相同的秩：
$$R \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{pmatrix} = R \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \\ \beta \end{pmatrix} = r.$$

设  $\xi_1, \xi_2, \dots, \xi_r$  是  $\alpha_1, \alpha_2, \dots, \alpha_m$  的一个极大无关组，则  $\xi_1, \xi_2, \dots, \xi_r, \beta$  是向量组  $\alpha_1, \alpha_2, \dots, \alpha_m, \beta$  中的  $r+1$  个向量，因而是线性相关的。所以  $\beta$  可由  $\xi_1, \xi_2, \dots, \xi_r$  线性表示，从而  $\beta$  可由  $\alpha_1, \alpha_2, \dots, \alpha_m$  线性表示。

**定理 2.1** 若向量组  $\alpha_1, \alpha_2, \dots, \alpha_m$  线性无关，而向量组  $\alpha_1, \alpha_2, \dots, \alpha_m, \beta$  线性相关，则向量  $\beta$  可以由向量组  $\alpha_1, \alpha_2, \dots, \alpha_m$  线性表示。

**2.11** 2.11 证明方程组 
$$\begin{cases} x_1 - x_2 = a_1 \\ x_2 - x_3 = a_2 \\ x_3 - x_4 = a_3 \\ x_4 - x_5 = a_4 \\ x_5 - x_1 = a_5 \end{cases}$$
 有解的充要条件是  $\sum_{i=1}^5 a_i = 0$ 。

$$\bar{A} = \left( \begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 0 & a_1 \\ 0 & 1 & -1 & 0 & 0 & a_2 \\ 0 & 0 & 1 & -1 & 0 & a_3 \\ 0 & 0 & 0 & 1 & -1 & a_4 \\ -1 & 0 & 0 & 0 & 1 & a_5 \end{array} \right) \rightarrow \left( \begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 0 & a_1 \\ 0 & 1 & -1 & 0 & 0 & a_2 \\ 0 & 0 & 1 & -1 & 0 & a_3 \\ 0 & 0 & 0 & 1 & -1 & a_4 \\ 0 & 0 & 0 & 0 & 0 & \sum_{i=1}^5 a_i \end{array} \right), \text{ 方程组有解} \Leftrightarrow R(\bar{A}) = R(A) = 4 \Leftrightarrow \sum_{i=1}^5 a_i = 0.$$

**2.12** 2.12 填空题

(1) 设  $\alpha_1 = (1, x, 1), \alpha_2 = (2, -1, 2), \alpha_3 = (0, 1, 2)$ ，当  $x = -\frac{1}{2}$  时  $\alpha_1, \alpha_2, \alpha_3$  线性相关。

(2) 当  $x = -1$  时，向量  $(x, 1, 0)$  能由下列向量组线性表示。

$$\alpha_1 = (1, -1, 0), \alpha_2 = (2, 0, -1)$$

(3) 已知向量组

$$\alpha_1 = (2, -1, 3, 0), \alpha_2 = (3, -1, 0, 1), \alpha_3 = (x, 0, -3, 1)$$

的秩等于 2，则  $x = 1$ 。

(4) 设矩阵  $A = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 2 & -1 & 2 & 1 \\ 1 & 2 & 0 & 2 \\ 2 & -1 & 3 & x \end{pmatrix}$ ，当  $x = 2$  时， $R(A) = 3$ 。

(5) 设  $\alpha_1, \alpha_2, \dots, \alpha_r$  是非齐次线性方程组  $AX = b$  的解，若  $k_1\alpha_1 + k_2\alpha_2 + \dots + k_r\alpha_r$  也是  $AX = b$  的一个解，则  $k_1 + k_2 + \dots + k_r = 1$ 。

(6) 设  $\alpha_1 = (2, 0, -1)^T, \alpha_2 = (1, 2, 0)^T$  是齐次线性方程组  $AX = 0$  的一个基础解系，则  $A = \begin{pmatrix} 2 & -1 & 4 \end{pmatrix}$ 。

$$(1) (\mathbf{a}_1^T, \mathbf{a}_2^T, \mathbf{a}_3^T) = \begin{pmatrix} 1 & 2 & 0 \\ x & -1 & 1 \\ 1 & 2 & 2 \end{pmatrix} \xrightarrow[r_3-r_1]{r_2-xr_1} \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1-2x & 1 \\ 0 & 0 & 2 \end{pmatrix} \xrightarrow[r_2-r_3]{\frac{1}{2}r_3} \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1-2x & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

当  $x = -\frac{1}{2}$  时,  $R\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\} = 2 < n = 3$ ,  $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$  线性相关.

(2) 设有  $x_1, x_2$  使得  $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = (x, 1, 0)$ , 即  $(\mathbf{a}_1^T, \mathbf{a}_2^T) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x \\ 1 \\ 0 \end{pmatrix}$ , 则该方程组的增广矩阵

$$\bar{A} = \left( \begin{array}{cc|c} 1 & 2 & x \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{array} \right) \rightarrow \left( \begin{array}{cc|c} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & x+1 \end{array} \right) \text{的秩 } R(\bar{A}) = R(A) = 2. \text{ 于是 } x = -1.$$

$$(3) (\mathbf{a}_1^T, \mathbf{a}_2^T, \mathbf{a}_3^T) = \begin{pmatrix} 2 & 3 & x \\ -1 & -1 & 0 \\ 3 & 0 & -3 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow[r_2+\frac{1}{2}r_1]{r_3+3r_2} \begin{pmatrix} 2 & 3 & x \\ 0 & \frac{1}{2} & \frac{x}{2} \\ 0 & -3 & -3 \\ 0 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 3 & x \\ 0 & 0 & x-1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \text{ 当 } R\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\} = 2 \text{ 时, } x = 1.$$

$$(4) A = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 2 & -1 & 2 & 1 \\ 1 & 2 & 0 & 2 \\ 2 & -1 & 3 & x \end{pmatrix} \xrightarrow[r_3-r_1]{r_3-r_2, r_4-2r_1} \begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & -5 & 4 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & x-1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & -5 & 4 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & x-2 \end{pmatrix}, \text{ 当 } x = 2 \text{ 时, } R(A) = 3$$

$$(5) \mathbf{b} = A(k_1\mathbf{a}_1 + k_2\mathbf{a}_2 + \cdots + k_r\mathbf{a}_r) = k_1A\mathbf{a}_1 + k_2A\mathbf{a}_2 + \cdots + k_rA\mathbf{a}_r = k_1\mathbf{b}_1 + k_2\mathbf{b}_2 + \cdots + k_r\mathbf{b}_r = (k_1 + k_2 + \cdots + k_r)\mathbf{b}$$

给出的是非齐次方程组,  $\mathbf{b} \neq 0$ , 所以  $k_1 + k_2 + \cdots + k_r = 1$

(6)  $m \times n$  线性方程组的基础解系含  $n-r$  个解向量 ( $r$  是系数矩阵  $A$  的秩). 现  $n=3, n-r=2$ , 于是  $r=1$ , 这说明方程组有一个有效

方程.  $A$  可以是一个行矩阵, 设为  $A = (a, b, c)$ . 因为  $\mathbf{a}_1 = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}, \mathbf{a}_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$  是方程组的解, 所以  $\begin{cases} 2a-c=0 \\ a+2b=0 \end{cases}$ . 解出  $a, b, c$  即知

$$A = (2 \ -1 \ 4).$$

注意: 本小题答案不是唯一的.

### 2.13 2.13 选择题

(1) 设向量组  $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$  线性无关, 则下列向量组线性相关的是 都不可选.

- (A)  $\mathbf{a}_1, \mathbf{a}_1 + \mathbf{a}_2, \mathbf{a}_1 + \mathbf{a}_3$       (B)  $\mathbf{a}_1, \mathbf{a}_1 + \mathbf{a}_2, \mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3$   
 (C)  $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3$       (D)  $\mathbf{a}_1 + \mathbf{a}_2, \mathbf{a}_1 + \mathbf{a}_3, \mathbf{a}_3 - \mathbf{a}_1$

(2) 在齐次线性方程组  $AX=0$  中, 若  $R(A) < n$ , 则下列结论正确的是 (A).

- (A) 当  $m=n$  时,  $A$  的  $m$  个行向量线性相关.  
 (B) 当  $m < n$  时,  $A$  的  $m$  个行向量线性无关.  
 (C) 当  $m > n$  时,  $A$  的  $m$  个行向量线性无关.  
 (D) 当  $n = m+1$  时,  $A$  的  $m$  个行向量线性相关.

(3) 设向量组  $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$  线性无关, 向量组  $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4$  线性相关, 则下列结论错误的是 (D).

- (A)  $\mathbf{a}_1, \mathbf{a}_2$  线性无关.      (B)  $\mathbf{a}_4$  可以表示为  $\mathbf{a}_1, \mathbf{a}_2$  线性组合.  
 (C)  $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$  线性相关. (D)  $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$  线性无关.

(4) 若非齐次线性方程组  $A_{m \times n} X = b$  有解,  $\alpha_1, \alpha_2, \dots, \alpha_n$  是  $A_{m \times n}$  的  $n$  个列向量, 下列结论正确的是 (A).

(A)  $\alpha_1, \alpha_2, \dots, \alpha_n, b$  线性相关. (B)  $\alpha_1, \alpha_2, \dots, \alpha_n$  线性无关.

(C)  $\alpha_1, \alpha_2, \dots, \alpha_n$  线性相关. (D)  $\alpha_1, \alpha_2, \dots, \alpha_n, b$  线性无关.

(5) 已知  $\beta_1, \beta_2$  是非齐次线性方程组  $AX = b$  的两个不同的解,  $\alpha_1, \alpha_2$  是对应的齐次线性方程组  $AX = 0$  的基础解系,  $k_1, k_2$  为任意常数, 则方程组  $AX = b$  的通解是 (B).

$$(A) k_1 \alpha_1 + k_2 (\alpha_1 + \alpha_2) + \frac{\beta_1 - \beta_2}{2} \quad (B) k_1 \alpha_1 + k_2 (\alpha_1 - \alpha_2) + \frac{\beta_1 + \beta_2}{2}$$

$$(C) k_1 \alpha_1 + k_2 (\beta_1 + \beta_2) + \frac{\beta_1 - \beta_2}{2} \quad (D) k_1 \alpha_1 + k_2 (\beta_1 - \beta_2) + \frac{\beta_1 + \beta_2}{2}$$

(6) 设  $A$  为  $n$  阶方阵, 且  $R(A) = n-1, \alpha_1, \alpha_2$  是  $AX = 0$  的两个不同的解向量, 则  $AX = 0$  的通解为 (C).

$$(A) k\alpha_1 \quad (B) k\alpha_2 \quad (C) k(\alpha_1 - \alpha_2) \quad (D) k(\alpha_1 + \alpha_2)$$

(1) 没有一个可选.

(A) 不是线性相关的, 因为

$$k_1 \alpha_1 + k_2 (\alpha_1 + \alpha_2) + k_3 (\alpha_1 + \alpha_3) = (k_1 + k_2 + k_3) \alpha_1 + k_2 \alpha_2 + k_3 \alpha_3 = 0 \Rightarrow \begin{cases} k_1 + k_2 + k_3 = 0 \\ k_2 = 0 \\ k_3 = 0 \end{cases} \Rightarrow \begin{cases} k_1 = 0 \\ k_2 = 0 \\ k_3 = 0 \end{cases}$$

(B) 不是线性相关的, 因为

$$k_1 \alpha_1 + k_2 (\alpha_1 + \alpha_2) + k_3 (\alpha_1 + \alpha_2 + \alpha_3) = (k_1 + k_2 + k_3) \alpha_1 + (k_2 + k_3) \alpha_2 + k_3 \alpha_3 = 0 \Rightarrow \begin{cases} k_1 + k_2 + k_3 = 0 \\ k_2 + k_3 = 0 \\ k_3 = 0 \end{cases} \Rightarrow \begin{cases} k_1 = 0 \\ k_2 = 0 \\ k_3 = 0 \end{cases}$$

(C) 不是线性相关的, 因为

$$k_1 \alpha_1 + k_2 \alpha_2 + k_3 (\alpha_1 + \alpha_2 + \alpha_3) = (k_1 + k_3) \alpha_1 + (k_2 + k_3) \alpha_2 + k_3 \alpha_3 = 0 \Rightarrow \begin{cases} k_1 + k_3 = 0 \\ k_2 + k_3 = 0 \\ k_3 = 0 \end{cases} \Rightarrow \begin{cases} k_1 = 0 \\ k_2 = 0 \\ k_3 = 0 \end{cases}$$

(D) 不是线性相关的, 因为

$$k_1 (\alpha_1 + \alpha_2) + k_2 (\alpha_1 + \alpha_3) + k_3 (\alpha_3 - \alpha_1) = (k_1 + k_2 - k_3) \alpha_1 + k_1 \alpha_2 + (k_2 + k_3) \alpha_3 = 0 \Rightarrow \begin{cases} k_1 + k_2 - k_3 = 0 \\ k_1 = 0 \\ k_2 + k_3 = 0 \end{cases} \Rightarrow \begin{cases} k_1 = 0 \\ k_2 = 0 \\ k_3 = 0 \end{cases}$$

(2) 设  $A_{m \times n} x_{n \times 1} = 0_{m \times 1}$ , 有  $R(A_{m \times n}) < n, m = n$ . 于是  $R(A_{m \times n}) < m$ . 故选  $A$  的  $m$  个行向量线性相关.

(3) 由  $\alpha_1, \alpha_2, \alpha_3$  线性无关可知  $\alpha_1, \alpha_2$  线性无关. 故(A)不可选.

由  $\alpha_1, \alpha_2, \alpha_4$  线性相关且  $\alpha_1, \alpha_2$  线性无关可知  $\alpha_4$  可由  $\alpha_1, \alpha_2$  线性表示 (定理 2.1 若向量组  $\alpha_1, \alpha_2, \dots, \alpha_m$  线性无关, 而向量组  $\alpha_1, \alpha_2, \dots, \alpha_m, \beta$  线性相关. 则向量  $\beta$  可以由向量组  $\alpha_1, \alpha_2, \dots, \alpha_m$  线性表示.) 故(B)不可选.

由  $\alpha_1, \alpha_2, \alpha_4$  线性相关可知  $\alpha_1, \alpha_2, \alpha_3, \alpha_4$  线性相关. 故(C)不可选.

上面确认了  $\alpha_1, \alpha_2, \alpha_3, \alpha_4$  是线性相关, 故选(D).

(4) 注意到  $A_{m \times n} x_{n \times 1} = b_{m \times 1} \Leftrightarrow x_1 \alpha_1 + x_2 \alpha_2 + \dots + x_n \alpha_n = b$ , 可知(A)成立.

(5) 由  $\alpha_1, \alpha_2$  是齐次方程组  $AX = 0$  的基础解系知  $AX = 0$  的基础解系一定含有两个线性无关的解向量(这两个向量一定是非零向量).

向量组  $\alpha_1, \alpha_1 + \alpha_2$  是线性无关的, 可作成一基础解系. 但  $\frac{\beta_1 - \beta_2}{2}$  不是非齐次方程组  $AX = b$  的特解, 因为

$$A\left(\frac{\beta_1 - \beta_2}{2}\right) = \frac{1}{2}(A\beta_1 - A\beta_2) = \frac{1}{2}(\mathbf{b} - \mathbf{b}) = \mathbf{0} \neq \mathbf{b}. \text{ 故不可选(A)}$$

$\alpha_1, \alpha_1 - \alpha_2$  线性无关故可作成一基础解系. 且  $\frac{\beta_1 + \beta_2}{2}$  是非齐次方程组  $AX = \mathbf{b}$  的一个特解, 因为

$$A\left(\frac{\beta_1 + \beta_2}{2}\right) = \frac{1}{2}(A\beta_1 + A\beta_2) = \frac{1}{2}(\mathbf{b} + \mathbf{b}) = \mathbf{b}. \text{ 故选 (B).}$$

$\beta_1 + \beta_2$  不是齐次方程组  $AX = 0$  的解, 因为  $A(\beta_1 + \beta_2) = A\beta_1 + A\beta_2 = \mathbf{b} + \mathbf{b} = 2\mathbf{b} \neq \mathbf{0}$ . 故不可选(C).

不能保证  $\alpha_1$  与  $\beta_1 - \beta_2$ , 故不选(D).

(6) 基础解系含有  $n - R(A) = n - (n-1) = 1$  个线性无关的解向量, 而任何一个单独的非零向量是线性无关的.

不能保证  $\alpha_1$  是非零向量, 故不选(A). 同理, 不选(B).  $\alpha_1, \alpha_2$  有可能是反方向的, 此时  $\alpha_1 = -\alpha_2$  或  $\alpha_1 + \alpha_2 = \mathbf{0}$ . 故不选(D). 由

$\alpha_1, \alpha_2$  是不同的解向量知  $\alpha_1 - \alpha_2 \neq \mathbf{0}$ , 因此可作成一基础解系. 故选 (C)

## 习题三

## 3.1 3.1 求下列矩阵的特征值和特征向量

$$(1) \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}; \quad (2) \begin{pmatrix} 5 & 6 & -3 \\ -1 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix}; \quad (3) \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix};$$

$$(4) \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}; \quad (5) \begin{pmatrix} 1 & 3 & 1 & 2 \\ 0 & -1 & 1 & 3 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

(1)解特征方程

$$|\lambda \mathbf{E} - \mathbf{A}| = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} \stackrel{r_2+r_1}{=} \begin{vmatrix} \lambda - 2 & -1 \\ \lambda - 3 & \lambda - 3 \end{vmatrix} \stackrel{r_1+r_2}{=} (\lambda - 3) \begin{vmatrix} \lambda - 2 & -1 \\ 1 & 1 \end{vmatrix} = (\lambda - 3)(\lambda - 1) = 0,$$

得特征值  $\lambda_1 = 1, \lambda_2 = 3$ .对于特征值  $\lambda_1 = 1$ , 解齐次线性方程组  $(\lambda \mathbf{E} - \mathbf{A})x = 0$ . 其系数矩阵

$$\lambda \mathbf{E} - \mathbf{A} = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},$$

可见特征向量为  $x = \begin{pmatrix} -k_1 \\ k_1 \end{pmatrix} = k_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} (k_1 \neq 0)$ .对于特征值  $\lambda_2 = 3$ ,  $\lambda \mathbf{E} - \mathbf{A} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$ , 可见特征向量为  $x = \begin{pmatrix} k_2 \\ k_2 \end{pmatrix} = k_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} (k_2 \neq 0)$ .

(2)解特征方程

$$|\lambda \mathbf{E} - \mathbf{A}| = \begin{vmatrix} \lambda - 5 & -6 & 3 \\ 1 & \lambda & -1 \\ -1 & -2 & \lambda - 1 \end{vmatrix} \stackrel{\substack{r_3+r_2 \\ r_1+3r_2}}{=} \begin{vmatrix} \lambda - 2 & 3\lambda - 6 & 0 \\ 1 & \lambda & -1 \\ 0 & \lambda - 2 & \lambda - 2 \end{vmatrix} = (\lambda - 2)^2 \begin{vmatrix} 1 & 3 & 0 \\ 1 & \lambda & -1 \\ 0 & 1 & 1 \end{vmatrix} \\ \stackrel{r_2-r_1+r_3}{=} (\lambda - 2)^2 \begin{vmatrix} 1 & 3 & 0 \\ 0 & \lambda - 2 & 0 \\ 0 & 1 & 1 \end{vmatrix} = (\lambda - 2)^3 \begin{vmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = (\lambda - 2)^3$$

得特征值  $\lambda = 2, 2, 2$ .对于特征值  $\lambda = 2, 2, 2$ , 解齐次线性方程组  $(\lambda \mathbf{E} - \mathbf{A})x = 0$ . 其系数矩阵

$$\lambda \mathbf{E} - \mathbf{A} = \begin{pmatrix} -3 & -6 & 3 \\ 1 & 2 & -1 \\ -1 & -2 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

可见特征向量为  $x = \begin{pmatrix} 2k_1 \\ -k_1 + k_2 \\ 2k_2 \end{pmatrix} = k_1 \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} (k_1, k_2 \text{ 不全为 } 0)$ .

(3)解特征方程



$$\begin{aligned}
 |\lambda E - A| &= \begin{vmatrix} \lambda-1 & -1 & -1 & -1 \\ -1 & \lambda-1 & 1 & 1 \\ -1 & 1 & \lambda-1 & 1 \\ -1 & 1 & 1 & \lambda-1 \end{vmatrix} \begin{matrix} r_2+r_1 \\ r_3+r_1 \\ r_4-r_1 \\ = \\ r_1+r_2+r_3+r_4 \end{matrix} \begin{vmatrix} \lambda-1 & -1 & -1 & -1 \\ \lambda-2 & \lambda-2 & 0 & 0 \\ \lambda-2 & 0 & \lambda-2 & 0 \\ \lambda-2 & 0 & 0 & \lambda-2 \end{vmatrix} \\
 &= (\lambda-2)^3 \begin{vmatrix} \lambda-1 & -1 & -1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix} = (\lambda-2)^3 \begin{vmatrix} \lambda+2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix} = (\lambda-2)^3 (\lambda+2)
 \end{aligned}$$

得特征值  $\lambda = -2, 2, 2, 2$ .

对于特征值  $\lambda = -2$ , 解齐次线性方程组  $(\lambda E - A)x = 0$ . 其系数矩阵

$$\lambda E - A = \begin{pmatrix} -3 & -1 & -1 & -1 \\ -1 & -3 & 1 & 1 \\ -1 & 1 & -3 & 1 \\ -1 & 1 & 1 & -3 \end{pmatrix} \begin{matrix} r_2+r_1 \\ r_3+r_1 \\ r_4+r_1 \\ \rightarrow \\ r_1+r_2+r_3+r_4 \end{matrix} \begin{pmatrix} -3 & -1 & -1 & -1 \\ -4 & -4 & 0 & 0 \\ -4 & 0 & -4 & 0 \\ -4 & 0 & 0 & -4 \end{pmatrix} \rightarrow \begin{pmatrix} -3 & -1 & -1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix},$$

可见特征向量为  $x = \begin{pmatrix} k_1 \\ -k_1 \\ -k_1 \\ -k_1 \end{pmatrix} = k_1 \begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \end{pmatrix} \quad (k_1 \neq 0)$ .

对于特征值  $\lambda = 2, 2, 2$ ,  $\lambda E - A = \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ .

可见特征向量为  $x = \begin{pmatrix} k_2 + k_3 + k_4 \\ k_2 \\ k_3 \\ k_4 \end{pmatrix} = k_2 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + k_3 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + k_4 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad (k_2, k_3, k_4 \text{ 不全为 } 0)$ .

$$(4) |\lambda E - A| = \begin{vmatrix} \lambda & 0 & -1 \\ 0 & \lambda-1 & 0 \\ -1 & 0 & \lambda \end{vmatrix} \begin{matrix} r_3+r_1 \\ = \\ r_3+r_1 \end{matrix} \begin{vmatrix} \lambda & 0 & -1 \\ 0 & \lambda-1 & 0 \\ \lambda-1 & 0 & \lambda-1 \end{vmatrix} = (\lambda-1)^2 \begin{vmatrix} \lambda & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = (\lambda-1)^2 (\lambda+1)$$

特征值  $\lambda_1 = -1, \lambda_2 = \lambda_3 = 1$ .

对于  $\lambda_1 = -1$ ,  $\lambda E - A = \begin{pmatrix} -1 & 0 & -1 \\ 0 & -2 & 0 \\ -1 & 0 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ , 特征向量  $x = k_1 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad (k_1 \neq 0)$ .

对于  $\lambda_2 = \lambda_3 = 1$ ,  $\lambda E - A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ . 特征向量  $x = k_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + k_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad (k_2, k_3 \text{ 不全为 } 0)$ .

$$(5) |\lambda E - A| = \begin{vmatrix} \lambda-1 & -3 & -1 & -2 \\ 0 & \lambda+1 & -1 & -3 \\ 0 & 0 & \lambda-2 & -5 \\ 0 & 0 & 0 & \lambda-2 \end{vmatrix} \begin{matrix} r_3+r_1 \\ = \\ r_3+r_1 \end{matrix} = (\lambda-1)(\lambda+1)(\lambda-2)^2$$

特征值  $\lambda_1 = -1, \lambda_2 = 1, \lambda_3 = \lambda_4 = 2$ .

$$\text{对于 } \lambda_1 = -1, \lambda E - A = \begin{pmatrix} -2 & -3 & -1 & -2 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & -3 & -5 \\ 0 & 0 & 0 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{特征向量 } x = k_1 \begin{pmatrix} -3 \\ 2 \\ 0 \\ 0 \end{pmatrix} (k_1 \neq 0).$$

$$\text{对于 } \lambda_2 = 1, \lambda E - A = \begin{pmatrix} 0 & -3 & -1 & -2 \\ 0 & 2 & -1 & -3 \\ 0 & 0 & -1 & -5 \\ 0 & 0 & 0 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \text{特征向量 } x = k_2 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} (k_2 \neq 0).$$

$$\text{对于 } \lambda_3 = \lambda_4 = 2, \lambda E - A = \begin{pmatrix} 1 & -3 & -1 & -2 \\ 0 & 3 & -1 & -3 \\ 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 3 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \text{特征向量 } x = k_3 \begin{pmatrix} 6 \\ 1 \\ 3 \\ 0 \end{pmatrix} (k_3 \neq 0).$$

**3.2** 3.2 已知矩阵  $A = \begin{bmatrix} 7 & 4 & -1 \\ 4 & 7 & -1 \\ -4 & -4 & x \end{bmatrix}$  的特征值为  $\lambda_1 = \lambda_2 = 3, \lambda_3 = 12$ , 求  $x$  的值, 并求其特征向量.

$$\lambda_1 + \lambda_2 + \lambda_3 = \text{tr}(A), \quad 3 + 3 + 12 = 7 + 7 + x, \quad x = 4$$

$$\text{对 } \lambda_1 = \lambda_2 = 3, \lambda E - A = \begin{pmatrix} 4 & 4 & -1 \\ 4 & 4 & -1 \\ -4 & -4 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 4 & 4 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad x = \begin{pmatrix} k_1 \\ k_2 \\ 4k_1 + 4k_2 \end{pmatrix} = k_1 \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} + k_2 \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} (k_1, k_2 \text{ 不全为 } 0).$$

注: 基础解系不是唯一的.

$$\text{对 } \lambda_3 = 12, \lambda E - A = \begin{pmatrix} 5 & -4 & 1 \\ -4 & 5 & 1 \\ 4 & 4 & 8 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad x = \begin{pmatrix} k_3 \\ k_3 \\ -k_3 \end{pmatrix} = k_3 \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} (k_3 \neq 0).$$

**3.3** 3.3 已知  $B = A^5 - 3A^4 + 2A - E$ , 其中  $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ , 试求  $B$  的特征值及  $|B|$ .

$$|\lambda A - E| = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = \lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3) = 0$$

得  $A$  的特征值  $\lambda = 1, 3$ .

$$B = \varphi(A) = A^5 - 3A^4 + 2A - E \text{ 的特征值 } \varphi(\lambda) = \lambda^5 - 3\lambda^4 + 2\lambda - 1: \varphi(1) = -1, \varphi(3) = -5. \text{ 故 } |B| = (-1) \cdot 5 = -5.$$

**3.4** 3.4 已知  $n$  阶矩阵  $A$  的特征值为  $2, 4, \dots, 2n$ , 试求  $|A - 3E|$ .

设  $\lambda$  是  $A$  的特征值, 则  $f(\lambda) = \lambda - 3$  是  $f(A) = A - 3E$  的特征值.

$$f(A) = A - 3E \text{ 的特征值有 } f(2) = -1, f(4) = 1, \dots, f(2n) = 2n - 3. \text{ 于是 } |A - 3E| = (-1) \cdot 1 \cdots (2n - 3)$$

**3.5** 3.5 试证: 若  $A^2 - 3A + 2E = 0$ , 则  $A$  的特征值只能是 1 或 2.

设  $\lambda$  是  $A$  的特征值, 则  $f(\lambda) = \lambda^2 - 3\lambda + 2$  是  $f(A) = A^2 - 3A + 2E$  的特征值. 现  $f(A) = 0$ , 而零矩阵的特征值都是 0. 于是  $A$

的特征值  $\lambda$  必须满足  $f(\lambda) = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2) = 0$ , 即  $\lambda = 1$  或  $\lambda = 2$ .

**3.6** 3.6 设  $A^2 = E$ , 试证  $A$  的特征值只能是 1 或 -1.

$$\varphi(A) = A^2 - E = 0 \text{ 的特征值满足 } \varphi(\lambda) = \lambda^2 - 1 = 0. \text{ 于是 } \lambda = \pm 1.$$

**3.7** 3.7 设  $\alpha_1, \alpha_2$  分别是  $A$  对应于  $\lambda_1, \lambda_2$  的特征向量,  $\lambda_1 \neq \lambda_2$ , 试证  $\alpha_1 + \alpha_2$  不可能是  $A$  的特征向量.

用反证法. 假如  $\alpha_1 + \alpha_2$  不可能是  $A$  的特征向量, 则有  $\lambda$  使得  $A(\alpha_1 + \alpha_2) = \lambda(\alpha_1 + \alpha_2)$ . 但  $\alpha_1, \alpha_2$  分别是  $A$  对应于  $\lambda_1, \lambda_2$  的特征向量,

有  $A\alpha_1 = \lambda_1\alpha_1, A\alpha_2 = \lambda_2\alpha_2$ . 从而  $A(\alpha_1 + \alpha_2) = A\alpha_1 + A\alpha_2 = \lambda_1\alpha_1 + \lambda_2\alpha_2$

**定理 3.2 推论 1:** 若  $n$  阶方阵  $A$  有互不相同的特征值  $\lambda_1, \lambda_2, \dots, \lambda_m$ . 则其对应的特征向量  $x_1, x_2, \dots, x_m$  线性无关.

于是有  $\lambda_1\alpha_1 + \lambda_2\alpha_2 = \lambda(\alpha_1 + \alpha_2)$  或  $(\lambda_1 - \lambda)\alpha_1 + (\lambda_2 - \lambda)\alpha_2 = 0$ . 但  $\alpha_1, \alpha_2$  是对应于不同特征值的特征向量, 因而是线性无关的, 故

$\lambda_1 - \lambda = \lambda_2 - \lambda = 0$  或  $\lambda_1 = \lambda_2 = \lambda$ ; 这是不可能的, 因为  $\lambda_1 \neq \lambda_2$ . 所以  $\alpha_1 + \alpha_2$  不可能是  $A$  的特征向量.

**3.8** 3.8 判断下列矩阵是否相似

$$(1) A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}, B = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix};$$

$$(2) A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

(1) 显然  $B$  的特征值是  $\lambda_1 = \lambda_2 = \lambda_3 = 3$ .

由  $\lambda E - B = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$  知  $B$  的特征向量为  $x = k \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} (k \neq 0)$ .  $B$  只有一个线性无关的特征向量, 因此  $B$  不可能与对角阵  $A$  相似.

**P73 定理 3.6**  $n$  阶矩阵  $A$  与  $n$  阶对角阵相似的充分必要条件是  $A$  有  $n$  个线性无关的特征向量.

(2) 显然  $B$  有 3 个相异的特征值  $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$ , 因此  $B$  与  $A$  相似.

**P74 定理 3.6 推论 3.2** 若  $n$  阶矩阵  $A$  有  $n$  个相异的特征值, 则  $A$  与对角阵相似.

**3.9** 3.9 设  $A = \begin{bmatrix} -2 & 0 & 0 \\ 2 & x & 2 \\ 3 & 1 & 1 \end{bmatrix}$ ,  $B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & y \end{bmatrix}$  且矩阵  $A$  和  $B$  相似, 试求: (1)  $x$  和  $y$  的值. (2) 可逆矩阵  $P$ , 使  $P^{-1}AP = B$ .

**P73 性质 3.2** 若  $n$  阶方阵  $A$  与  $B$  相似, 则 (1)  $|A| = |B|$ , (2)  $\text{tr} A = \text{tr} B$ .

相似矩阵  $A$  和  $B$  有相同的行列式和迹:  $\begin{vmatrix} -2 & 0 & 0 \\ 2 & x & 2 \\ 3 & 1 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & y \end{vmatrix}$ ,  $-2 + x + 1 = -1 + 2 + y$ .

都得到  $x = y + 2$ . 相似矩阵  $A$  和  $B$  有相同的特征多项式

$$|\lambda E - A| = (\lambda + 2)((\lambda - x)(\lambda - 1) - 2), \quad |\lambda E - B| = (\lambda + 1)(\lambda - 2)(\lambda - y).$$

比较得  $\lambda + 2 = \lambda - y$  或  $y = -2$ . 而  $x = y + 2 = 0$ .

矩阵  $A$  的特征值为  $\lambda_1 = -2, \lambda_2 = -1, \lambda_3 = 2$ .

对于  $\lambda_1 = -2, \lambda E - A = \begin{bmatrix} 0 & 0 & 0 \\ -2 & -2 & -2 \\ -3 & -1 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ , 得特征向量  $k_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} (k_1 \neq 0)$ .

对于  $\lambda_2 = -1, \lambda E - A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & -1 & -2 \\ -3 & -1 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ , 得特征向量  $k_2 \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} (k_2 \neq 0)$ .

对于  $\lambda_3 = 2$ ,  $\lambda E - A = \begin{bmatrix} 4 & 0 & 0 \\ -2 & 2 & -2 \\ -3 & -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ , 得特征向量  $k_3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$  ( $k_3 \neq 0$ ).

令  $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ -1 & -1 & 1 \end{pmatrix}$ , 则有  $P^{-1}AP = B$ .

**3.10** 3.10 将下列矩阵对角化, 并求  $P$ , 使  $P^{-1}AP = A$  ( $A$  为对角阵)

$$(1) A = \begin{bmatrix} 4 & 6 & 0 \\ -3 & -5 & 0 \\ -3 & -6 & 1 \end{bmatrix} \quad (2) A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 2 & 1 & 2 \end{bmatrix}$$

$$(3) A = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (4) A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(1) 解特征方程  $|\lambda E - A| = \begin{vmatrix} \lambda-4 & -6 & 0 \\ 3 & \lambda+5 & 0 \\ 3 & 6 & \lambda-1 \end{vmatrix} \begin{matrix} r_2+r_1 \\ r_3+r_1 \end{matrix} = (\lambda-1)^2 \begin{vmatrix} \lambda-4 & -6 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = (\lambda-1)^2(\lambda+2)$ , 得特征值

$$\lambda_1 = -2, \lambda_2 = \lambda_3 = 1.$$

对于  $\lambda_1 = -2$ ,  $\lambda E - A = \begin{bmatrix} -6 & -6 & 0 \\ 3 & 3 & 0 \\ 3 & 6 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ , 得特征向量  $k_1 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$  ( $k_1 \neq 0$ ). 选  $\alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ .

对于  $\lambda_2 = \lambda_3 = 1$ ,  $\lambda E - A = \begin{bmatrix} -3 & -6 & 0 \\ 3 & 6 & 0 \\ 3 & 6 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ , 得特征向量  $k_2 \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + k_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$  ( $k_2, k_3$  不全为 0). 选  $\alpha_2 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ .

令  $P = (\alpha_1, \alpha_2, \alpha_3) = \begin{pmatrix} -1 & 2 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ , 则有  $P^{-1}AP = A = \begin{pmatrix} -2 & & \\ & 1 & \\ & & 1 \end{pmatrix}$ .

(2) 解特征方程  $|\lambda E - A| = \begin{vmatrix} \lambda-1 & -2 & -3 \\ 0 & \lambda-1 & 0 \\ -2 & -1 & \lambda-2 \end{vmatrix} \begin{matrix} r_1+2r_2 \\ r_3+r_2 \end{matrix} = (\lambda-1) \begin{vmatrix} \lambda-1 & 0 & -3 \\ 0 & 1 & 0 \\ -2 & 0 & \lambda-2 \end{vmatrix} = (\lambda-1)[(\lambda-1)(\lambda-2)-6] = 0$ , 得特征值

$$\lambda_1 = -1, \lambda_2 = 1, \lambda_3 = 4.$$

对于  $\lambda_1 = -1$ ,  $\lambda E - A = \begin{bmatrix} -2 & -2 & -3 \\ 0 & -2 & 0 \\ -2 & -1 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ , 得特征向量  $k_1 \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix}$  ( $k_1 \neq 0$ ). 选  $\alpha_1 = \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix}$ .

对于  $\lambda_2 = 1$ ,  $\lambda E - A = \begin{bmatrix} 0 & -2 & -3 \\ 0 & 0 & 0 \\ -2 & -1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 4 & 0 & -1 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ , 得特征向量  $k_2 \begin{pmatrix} 1 \\ -6 \\ 4 \end{pmatrix}$  ( $k_2 \neq 0$ ). 选  $\alpha_2 = \begin{pmatrix} 1 \\ -6 \\ 4 \end{pmatrix}$ .

对于  $\lambda_3 = 4$ ,  $\lambda E - A = \begin{bmatrix} 3 & -2 & -3 \\ 0 & 3 & 0 \\ -2 & -1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ , 得特征向量  $k_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$  ( $k_3 \neq 0$ ). 选  $\alpha_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ .

令  $P = (\alpha_1, \alpha_2, \alpha_3) = \begin{pmatrix} 3 & 1 & 1 \\ 0 & -6 & 0 \\ -2 & 4 & 1 \end{pmatrix}$ , 则有  $P^{-1}AP = A = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 4 \end{pmatrix}$ .

**3.11** 3.11 设  $A = \begin{bmatrix} 0 & 2 \\ -3 & 5 \end{bmatrix}$ , 求  $A^{100}$ .

$|\lambda E - A| = \begin{vmatrix} \lambda & -2 \\ 3 & \lambda - 5 \end{vmatrix} = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3)$ , 得特征值  $\lambda_1 = 2, \lambda_2 = 3$ .

对于  $\lambda_1 = 2$ ,  $\lambda E - A = \begin{pmatrix} 2 & -2 \\ 3 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$ , 选  $\alpha_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ .

对于  $\lambda_2 = 3$ ,  $\lambda E - A = \begin{pmatrix} 3 & -2 \\ 3 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 3 & -2 \\ 0 & 0 \end{pmatrix}$ , 选  $\alpha_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ .

令  $P = (\alpha_1, \alpha_2) = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$ , 则有  $P^{-1}AP = A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ . 于是  $A = PAP^{-1}$ , 而

$$A^{100} = PA^{100}P^{-1} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 2^{100} & 0 \\ 0 & 3^{100} \end{pmatrix} \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 3 \cdot 2^{100} - 2 \cdot 3^{100} & -2 \cdot 2^{100} + 2 \cdot 3^{100} \\ 3 \cdot 2^{100} - 3 \cdot 3^{100} & -2 \cdot 2^{100} + 3 \cdot 3^{100} \end{pmatrix} = \begin{pmatrix} 3 \cdot 2^{100} - 2 \cdot 3^{100} & 2 \cdot 3^{100} - 2^{101} \\ 3 \cdot 2^{100} - 3^{101} & 3^{101} - 2^{101} \end{pmatrix}$$

**3.12** 3.12 试求一个正交矩阵  $P$ , 使  $P^{-1}AP = A$

$$(1) A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad (2) A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

(1) 解特征方程  $|\lambda E - A| = \begin{vmatrix} \lambda - 1 & -1 & -1 \\ -1 & \lambda - 1 & -1 \\ -1 & -1 & \lambda - 1 \end{vmatrix} \xrightarrow[r_3 - r_2]{r_2 - r_1} \begin{vmatrix} \lambda - 1 & -1 & -1 \\ -\lambda & \lambda & 0 \\ -\lambda & 0 & \lambda \end{vmatrix} \xrightarrow[r_1 + r_2 + r_3]{\frac{1}{\lambda} r_2, \frac{1}{\lambda} r_3} \lambda^2 \begin{vmatrix} \lambda - 3 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = \lambda^2(\lambda - 3)$ , 得特征值

$\lambda_1 = \lambda_2 = 0, \lambda_3 = 3$ .

对于  $\lambda_1 = \lambda_2 = 0$ ,  $\lambda E - A = \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ , 得特征向量  $k_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + k_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$  ( $k_1, k_2$  不全为 0). 选

$$\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

将  $\alpha_1, \alpha_2$  正交化:  $\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \beta_2 = \alpha_2 - \frac{[\alpha_2, \beta_1]}{[\beta_1, \beta_1]} \beta_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$ .

对于  $\lambda_3 = 3$ ,  $\lambda E - A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ , 得特征向量  $k_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$  ( $k_3 \neq 0$ ). 选  $\beta_3 = \alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ .

将  $\beta_1, \beta_2, \beta_3$  单位化:  $\gamma_1 = \frac{1}{\|\beta_1\|} \beta_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \gamma_2 = \frac{1}{\|\beta_2\|} \beta_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}, \gamma_3 = \frac{1}{\|\beta_3\|} \beta_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

令  $P = (\gamma_1, \gamma_2, \gamma_3) = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}$ , 则  $P$  是正交的, 且  $P^{-1}AP = A = \begin{pmatrix} 0 & & \\ & 0 & \\ & & 3 \end{pmatrix}$ .

**3.13** \*3.13 证明任何一个矩阵能唯一写成一个对称阵与一个反对称阵之和.

任给矩阵  $A$ , 有  $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$ , 其中  $\frac{1}{2}(A + A^T)$  是对称矩阵:

$$\left(\frac{1}{2}(A + A^T)\right)^T = \frac{1}{2}(A^T + A) = \frac{1}{2}(A + A^T),$$

而  $\frac{1}{2}(A - A^T)$  是反对称矩阵:

$$\left(\frac{1}{2}(A - A^T)\right)^T = \frac{1}{2}(A^T - A) = -\frac{1}{2}(A - A^T).$$

故  $A$  能唯一写成一个对称阵与一个反对称阵之和.

又假设  $A$  可写成  $A = B + C$ , 其中  $B$  是对称的, 而  $C$  是反对称的. 则

$$A^T = B^T + C^T = B - C.$$

于是

$$A + A^T = 2B, \quad A - A^T = 2C, \quad \text{或} \quad B = \frac{1}{2}(A + A^T), \quad C = \frac{1}{2}(A - A^T).$$

可见  $A$  写成一个对称阵与一个反对称阵之和的形式是唯一的..

**3.14** 3.14 建模题

**3.15** 3.15 建模题

**3.16** 3.16 填空题

(1) 设三阶方阵  $A$  的三个特征值为  $\lambda_1 = 2, \lambda_2 = -1, \lambda_3 = 3$ , 则  $A$  的伴随矩阵对应的行列式  $|A^*|$  为 36.

$$|A^*| = |A|^{n-1} = (2 \cdot (-1) \cdot 3)^2 = 36, \quad A^* = |A|A^{-1} = -6A^{-1} \text{ 的特征值为 } -6 \cdot \frac{1}{2} = -3, \quad -6 \cdot (-1) = 6, \quad -6 \cdot \frac{1}{3} = -2$$

(2) 设 0 是矩阵  $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & a \end{bmatrix}$  的特征值, 则  $a =$  1.

$$|A| = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & a \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & a-1 \end{vmatrix} = 1 \cdot 2 \cdot (a-1) = 0 \Rightarrow a = 1$$

(3) 设  $A$  为三阶方阵,  $\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = 2$  为三个特征值, 对应特征向量为  $\alpha_1, \alpha_2, \alpha_3$ , 令  $P = (\alpha_1 \ \alpha_2 \ \alpha_3)$ , 则

$$P^{-1}A^*P = \begin{pmatrix} -2 & & \\ & 2 & \\ & & -1 \end{pmatrix}.$$

$$\begin{aligned} P^{-1}A^*P &= P^{-1}(|A|A^{-1})P = |A|P^{-1}A^{-1}P = |A|(P^{-1}AP)^{-1} = |A|A^{-1} \\ &= \lambda_1\lambda_2\lambda_3 \begin{pmatrix} \lambda_1^{-1} & & \\ & \lambda_2^{-1} & \\ & & \lambda_3^{-1} \end{pmatrix} = \begin{pmatrix} \lambda_2\lambda_3 & & \\ & \lambda_1\lambda_3 & \\ & & \lambda_1\lambda_2 \end{pmatrix} = \begin{pmatrix} -2 & & \\ & 2 & \\ & & -1 \end{pmatrix} \end{aligned}$$

(4) 设  $A = \begin{bmatrix} a & 2 & 1 \\ 0 & 3 & 2 \\ 2 & 1 & -2 \end{bmatrix}$ ,  $B = \begin{bmatrix} b & 0 & 1 \\ 2 & 5 & 3 \\ 1 & 1 & 0 \end{bmatrix}$  有相同的特征值, 则  $a = \frac{-7}{5}$ ,  $b = \frac{-27}{5}$ .

$$|A| = \begin{vmatrix} a & 2 & 1 \\ 0 & 3 & 2 \\ 2 & 1 & -2 \end{vmatrix} = 2 - 8a = |B| = \begin{vmatrix} b & 0 & 1 \\ 2 & 5 & 3 \\ 1 & 1 & 0 \end{vmatrix} = -3 - 3b, \operatorname{tr} A = a + 3 - 2 = \operatorname{tr} B = b + 5, \text{ 解得 } a = -\frac{7}{5}, b = -\frac{27}{5}$$

(5)\* 已知四阶方阵  $A$  相似于  $B$ ,  $A$  的特征值为 2, 3, 4, 5,  $E$  为四阶单位矩阵, 则  $|B - E| = \underline{24}$ .

$B$  与  $A$  相似, 因此有相同的特征值, 从而  $B - E$  为  $\lambda - 1 = 1, 2, 3, 4$ . 于是  $|B - E| = 1 \cdot 2 \cdot 3 \cdot 4 = 24$

### 3.17 3.17 选择题

(1) 设  $\alpha_1, \alpha_2$  是  $A$  的对应于  $\lambda_0$  的两个不同特征向量, 则如下为  $A$  的特征向量的有 (D).

- (A)  $k\alpha_1$     (B)  $k\alpha_2$     (C)  $\alpha_1 + \alpha_2$     (D)  $\alpha_1 - \alpha_2$

$$A(\alpha_1 - \alpha_2) = A\alpha_1 - A\alpha_2 = \lambda_0\alpha_1 - \lambda_0\alpha_2 = \lambda_0(\alpha_1 - \alpha_2).$$

由于  $\alpha_1, \alpha_2$  是不同的,  $\alpha_1 - \alpha_2$  不是零向量, 故  $\alpha_1 - \alpha_2$  是  $A$  的特征向量, 且是属于特征值  $\lambda_0$  的. 故选 (D).

事实上, 属于同一特征值的特征向量的非零线性组合仍是属于该特征值的特征向量. (A), (B), (C) 中的向量有可能是零向量, 故不选 (A), (B), (C).

(2) 已知三阶矩阵  $A$  的特征值为 1, 2, -1, 则矩阵  $B = (2A^*)^{-1}$  (其中  $A^*$  为  $A$  的伴随阵) 的特征值为 (B).

- (A) -1, -2, 1    (B)  $-\frac{1}{4}, -\frac{1}{2}, \frac{1}{4}$     (C)  $\frac{1}{4}, -\frac{1}{2}, -\frac{1}{4}$     (D)  $-\frac{1}{2}, -1, \frac{1}{2}$

注意到  $|A| = 1 \cdot 2 \cdot (-1) = -2$ , 得  $B = (2A^*)^{-1} = (2|A|A^{-1})^{-1} = \frac{1}{2|A|}A$ .

设  $\lambda$  是  $A$  的特征值, 则  $\frac{1}{2|A|}\lambda = -\frac{1}{4}\lambda$  是  $B$  的特征值. 可知,  $B$  的特征值有  $-\frac{1}{4}, -\frac{2}{4}, -\frac{-1}{4}$  或  $-\frac{1}{4}, -\frac{1}{2}, \frac{1}{4}$ .

(3) 已知  $\lambda_1, \lambda_2, \lambda_3$  为三阶方阵  $A$  的三个不同的特征值, 对应的特征向量依次为  $\alpha_1, \alpha_2, \alpha_3$ , 令  $P = (-\alpha_1 \ 2\alpha_2 \ 3\alpha_3)$ , 则  $P^{-1}AP$  等于 (B).

- (A)  $\begin{bmatrix} -\lambda_1 & & \\ & 2\lambda_2 & \\ & & 3\lambda_3 \end{bmatrix}$     (B)  $\begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix}$

$$(C) \begin{bmatrix} \lambda_1 & & \\ & 2\lambda_2 & \\ & & 3\lambda_3 \end{bmatrix} \quad (D) \text{ 不能确定.}$$

$$P = (-\alpha_1 \ 2\alpha_2 \ 3\alpha_3) = (\alpha_1 \ \alpha_2 \ \alpha_3) \begin{pmatrix} -1 & & \\ & 2 & \\ & & 3 \end{pmatrix} = Q \begin{pmatrix} -1 & & \\ & 2 & \\ & & 3 \end{pmatrix},$$

其中  $Q = (\alpha_1 \ \alpha_2 \ \alpha_3)$

$$\begin{aligned} P^{-1}AP &= \begin{pmatrix} -1 & & \\ & 2 & \\ & & 3 \end{pmatrix}^{-1} (\alpha_1 \ \alpha_2 \ \alpha_3)^{-1} A (\alpha_1 \ \alpha_2 \ \alpha_3) \begin{pmatrix} -1 & & \\ & 2 & \\ & & 3 \end{pmatrix} \\ &= \begin{pmatrix} -1 & & \\ & 2 & \\ & & 3 \end{pmatrix}^{-1} \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} \begin{pmatrix} -1 & & \\ & 2 & \\ & & 3 \end{pmatrix} \\ &= \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} \end{aligned}$$

**P90 定理 3.6 推论 3.2** 若  $n$  阶矩阵  $A$  有  $n$  个相异的特征值, 则  $A$  与对角阵相似.

**解法二** 或者更简单地, 由  $\lambda_1, \lambda_2, \lambda_3$  的相异性知  $A$  可对角化. 特征向量  $\alpha_1, \alpha_2, \alpha_3$  分别属于  $\lambda_1, \lambda_2, \lambda_3$ , 而向量  $-\alpha_1, 2\alpha_2, 3\alpha_3$  同样是分别属于  $\lambda_1, \lambda_2, \lambda_3$  的特征向量, 因此所得矩阵  $P = (-\alpha_1 \ 2\alpha_2 \ 3\alpha_3)$  可作为相似对角变换的矩阵, 即

$$P^{-1}AP = A = \text{diag}(\lambda_1, \lambda_2, \lambda_3).$$

(4) 如果下列哪一个成立, 则  $A$  与  $B$  相似. **(D)**

(A)  $|A| = |B|$  (B)  $R(A) = R(B)$  (C)  $A$  与  $B$  有相同的特征值. (D)  $A$  与  $B$  有相同的特征值且  $n$  个特征值各不相同.

选(D). 这时候  $A$  和  $B$  都与同一个对角阵  $A = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$  相似, 其中  $\lambda_1, \lambda_2, \lambda_3$  是  $A$  和  $B$  的共同特征值.



## 习题四

4.1 4.1 设  $\alpha = [10, -8, 6, -5]^T$ ,  $\beta = [2, 2, 0, -5]^T$ , 求  $[\alpha, \alpha]$ ,  $[\alpha, \beta]$ ,  $[3\alpha - \beta, 2\alpha]$ ,  $\|\alpha\|$ ,  $\|\alpha - \beta\|$ .

$$[\alpha, \alpha] = 10 \cdot 10 + (-8) \cdot (-8) + 6 \cdot 6 + (-5) \cdot (-5) = 225,$$

$$[\alpha, \beta] = 10 \cdot 2 + (-8) \cdot 2 + 6 \cdot 0 + (-5) \cdot (-5) = 29,$$

$$[3\alpha - \beta, 2\alpha] = 6[\alpha, \alpha] - 2[\beta, \alpha] = 6[\alpha, \alpha] - 2[\alpha, \beta] = 6 \cdot 225 - 2 \cdot 29 = 1292,$$

$$\|\alpha\| = \sqrt{[\alpha, \alpha]} = \sqrt{225} = 15 \quad \|\alpha - \beta\| = \|[8, -10, 6, 0]\| = \sqrt{8^2 + (-10)^2 + 6^2 + 0^2} = 10\sqrt{2}$$

4.2 4.2 设有三点  $A(-1, 2, 1)$ ,  $B(0, 3, 1)$ ,  $C(0, 2, 2)$ , 求  $\angle BAC$ .

$$\overline{AB} = (0, 3, 1) - (-1, 2, 1) = (1, 1, 0)$$

$$\overline{AC} = (0, 2, 2) - (-1, 2, 1) = (1, 0, 1)$$

$$[\overline{AB}, \overline{AC}] = 1 \cdot 1 + 1 \cdot 0 + 0 \cdot 1 = 1$$

$$\|\overline{AB}\| = \sqrt{1^2 + 1^2 + 0^2} = \sqrt{2}, \|\overline{AC}\| = \sqrt{1^2 + 0^2 + 1^2} = \sqrt{2}$$

$$\angle BAC = \arccos \frac{[\overline{AB}, \overline{AC}]}{\|\overline{AB}\| \|\overline{AC}\|} = \arccos \frac{1}{\sqrt{2} \cdot \sqrt{2}} = \frac{\pi}{3}$$

4.3 4.3 设  $\alpha$  与  $\beta$  的夹角是  $120^\circ$ , 模分别为 12 和 6, 求  $\alpha$  与  $\beta$  的距离.

利用 p87 例 4.6 的结果,  $\alpha$  与  $\beta$  的距离为

$$\begin{aligned} \|\alpha - \beta\| &= \sqrt{\|\alpha\|^2 + \|\beta\|^2 - 2\|\alpha\| \cdot \|\beta\| \cos \theta} \\ &= \sqrt{12^2 + 6^2 - 2 \cdot 12 \cdot 6 \cdot \cos 120^\circ} = \sqrt{252} \approx 15.87 \end{aligned}$$

4.4 4.4 作为平面几何定理(平行四边形的对角线与边长的关系)的推广, 证明

$$\|\alpha + \beta\|^2 + \|\alpha - \beta\|^2 = 2(\|\alpha\|^2 + \|\beta\|^2).$$

$$\begin{aligned} \|\alpha + \beta\|^2 + \|\alpha - \beta\|^2 &= [\alpha + \beta, \alpha + \beta] + [\alpha - \beta, \alpha - \beta] \\ &= ([\alpha, \alpha] + [\alpha, \beta] + [\beta, \alpha] + [\beta, \beta]) + ([\alpha, \alpha] - [\alpha, \beta] - [\beta, \alpha] + [\beta, \beta]) \\ &= 2([\alpha, \alpha] + [\beta, \beta]) = 2(\|\alpha\|^2 + \|\beta\|^2) \end{aligned}$$

4.5 4.5 设  $\alpha_1 = (1, 2, -1)^T$ ,  $\alpha_2 = (-1, 3, 1)^T$ ,  $\alpha_3 = (4, -1, 0)^T$ , 试用施密特正交化方法把这组向量正交规范化.

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \quad \beta_2 = \alpha_2 - \frac{[\alpha_2, \beta_1]}{[\beta_1, \beta_1]} \beta_1 = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} - \frac{4}{6} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \frac{5}{3} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix},$$

正交化:

$$\beta_3 = \alpha_3 - \frac{[\alpha_3, \beta_1]}{[\beta_1, \beta_1]} \beta_1 - \frac{[\alpha_3, \beta_2]}{[\beta_2, \beta_2]} \beta_2 = \begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix} - \frac{2}{6} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} - \frac{-5}{3} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

$$\text{单位化: } \eta_1 = \frac{1}{\|\beta_1\|} \beta_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \quad \eta_2 = \frac{1}{\|\beta_2\|} \beta_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \quad \eta_3 = \frac{1}{\|\beta_3\|} \beta_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

4.6 4.6 已知  $\alpha_1 = (1, 1, 1)^T$ ,  $\alpha_2 = (1, -2, 1)^T$  正交, 试求一个非零向量  $\alpha_3$ , 使  $\alpha_1, \alpha_2, \alpha_3$  两两正交.

设  $\alpha_3 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ . 因为  $\alpha_3$  与  $\alpha_1, \alpha_2$  都正交:  $[\alpha_1, \alpha_3] = 0, [\alpha_2, \alpha_3] = 0$ , 所以  $\begin{cases} x_1 + x_2 + x_3 = 0 \\ x_1 - 2x_2 + x_3 = 0 \end{cases}$ .

由  $\begin{pmatrix} \alpha_1^T \\ \alpha_2^T \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ , 得  $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} k \\ 0 \\ -k \end{pmatrix} = k \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ . 不妨取  $\alpha_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ .

4.7 4.7 设  $A, B$  都是  $n$  阶正交矩阵, 证明  $AB$  也是正交矩阵.

$$(AB)(AB)^T = A(BB^T)A^T = AEA^T = AA^T = E$$

4.8 4.8 设  $\alpha_1 = [5, 3, 1, 1]^T$ ,  $\alpha_2 = [-4, 18, -2, 4]^T$ , 欲使向量  $\beta = \alpha_2 + k\alpha_1$  与  $\alpha_1$  正交, 求  $k$ .

由  $[\alpha_1, \beta] = [\alpha_1, \alpha_2 + k\alpha_1] = [\alpha_1, \alpha_2] + k[\alpha_1, \alpha_1] = 0$ , 得  $k = -\frac{[\alpha_1, \alpha_2]}{[\alpha_1, \alpha_1]} = -\frac{36}{36} = -1$

4.9 4.9 设  $\alpha_1, \dots, \alpha_s$  是正交向量组, 试证  $\|\alpha_1 + \dots + \alpha_s\|^2 = \|\alpha_1\|^2 + \dots + \|\alpha_s\|^2$ .

先证  $s=2$  的情形:

$$\begin{aligned} \|\alpha_1 + \alpha_2\|^2 &= [\alpha_1 + \alpha_2, \alpha_1 + \alpha_2] = [\alpha_1, \alpha_1] + [\alpha_1, \alpha_2] + [\alpha_2, \alpha_1] + [\alpha_2, \alpha_2] \\ &= \|\alpha_1\|^2 + 0 + 0 + \|\alpha_2\|^2 = \|\alpha_1\|^2 + \|\alpha_2\|^2. \end{aligned}$$

假设对于  $s-1$  个向量, 公式成立, 那么注意到  $\alpha_1 + \dots + \alpha_{s-1}$  与  $\alpha_s$  正交, 有

$$\begin{aligned} \|\alpha_1 + \dots + \alpha_s\|^2 &= \|(\alpha_1 + \dots + \alpha_{s-1}) + \alpha_s\|^2 = \|\alpha_1 + \dots + \alpha_{s-1}\|^2 + \|\alpha_s\|^2 \\ &= (\|\alpha_1\|^2 + \dots + \|\alpha_{s-1}\|^2) + \|\alpha_s\|^2 = \|\alpha_1\|^2 + \dots + \|\alpha_{s-1}\|^2 + \|\alpha_s\|^2 \end{aligned}$$

公式得证.

4.10 4.10 设  $\alpha_1, \alpha_2, \alpha_3, \alpha_4$  是正交向量组, 且  $\|\alpha_i\| = i, i=1, 2, 3, 4$ , 求  $\|\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4\|$ ; 记  $A = [\alpha_1, \alpha_2, \alpha_3, \alpha_4]$ , 求  $|A^T A|$ .

利用 4.9 题的结论, 得  $\|\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4\| = \sqrt{\|\alpha_1\|^2 + \|\alpha_2\|^2 + \|\alpha_3\|^2 + \|\alpha_4\|^2} = \sqrt{1^2 + 2^2 + 3^2 + 4^2} = \sqrt{30}$ .

$$\text{而 } |A^T A| = \begin{vmatrix} \alpha_1^T \\ \alpha_2^T \\ \alpha_3^T \\ \alpha_4^T \end{vmatrix} (\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \begin{vmatrix} \alpha_1^T \alpha_1 & \alpha_1^T \alpha_2 & \alpha_1^T \alpha_3 & \alpha_1^T \alpha_4 \\ \alpha_2^T \alpha_1 & \alpha_2^T \alpha_2 & \alpha_2^T \alpha_3 & \alpha_2^T \alpha_4 \\ \alpha_3^T \alpha_1 & \alpha_3^T \alpha_2 & \alpha_3^T \alpha_3 & \alpha_3^T \alpha_4 \\ \alpha_4^T \alpha_1 & \alpha_4^T \alpha_2 & \alpha_4^T \alpha_3 & \alpha_4^T \alpha_4 \end{vmatrix} = \begin{vmatrix} 1^2 & 0 & 0 & 0 \\ 0 & 2^2 & 0 & 0 \\ 0 & 0 & 3^2 & 0 \\ 0 & 0 & 0 & 4^2 \end{vmatrix} = 576.$$

4.11 4.11 设  $\alpha$  和  $\beta$  都是  $n$  维向量, 其中  $\alpha \neq 0$ , 试问  $k$  为何值时, 距离  $\|\beta - k\alpha\|$  取最小值?

令  $f(k) = \|\beta - k\alpha\|^2$ , 则  $f(k) = [\beta - k\alpha, \beta - k\alpha] = \|\beta\|^2 - 2k[\alpha, \beta] + k^2\|\alpha\|^2 = \left(k\|\alpha\| - \frac{[\alpha, \beta]}{\|\alpha\|}\right)^2 + \|\beta\|^2 - \frac{[\alpha, \beta]^2}{\|\alpha\|^2} \geq \|\beta\|^2 - \frac{[\alpha, \beta]^2}{\|\alpha\|^2}$ .

当  $k = \frac{[\alpha, \beta]}{\|\alpha\|^2}$  时,  $f(k) = \|\beta - k\alpha\|^2$  取得最小值  $\|\beta\|^2 - \frac{[\alpha, \beta]^2}{\|\alpha\|^2}$ , 而距离  $\|\beta - k\alpha\|$  取最小值  $\frac{\sqrt{\|\alpha\|^2 \|\beta\|^2 - [\alpha, \beta]^2}}{\|\alpha\|}$ .

4.12 4.12 设  $A = \frac{1}{19} \begin{pmatrix} 15 & 6 & -10 \\ 10 & -15 & 6 \\ 6 & 10 & 16 \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$ ,

(1) 验证线性变换  $y = Ax$ ,  $x \in \mathbb{R}^3$  是正交变换.

(2)  $x = [3, 2, 0]^T$ , 求  $y$ ; 设  $y = [5, 2, 0]^T$ , 求  $x$ .

(1) 因为下列各式成立

$$[\alpha_1, \alpha_2] = \frac{1}{19^2} [15 \times 10 + 6 \times (-15) + (-10) \times 6] = 0$$

$$[\alpha_1, \alpha_3] = \frac{1}{19^2} [15 \times 6 + 6 \times 10 + (-10) \times 16] = 0$$

$$[\alpha_2, \alpha_3] = \frac{1}{19^2} [10 \times 6 + (-15) \times 10 + 6 \times 16] = 0$$

$$[\alpha_1, \alpha_1] = \frac{1}{19^2} [15^2 + 6^2 + (-10)^2] = 1$$

$$[\alpha_2, \alpha_2] = \frac{1}{19^2} (6^2 + 10^2 + 16^2) = 1$$

$$[\alpha_3, \alpha_3] = \frac{1}{19^2} (6^2 + 10^2 + 16^2) = 1$$

即  $A$  的行向量组是标准正交向量组, 所以  $A$  是正交矩阵.

$$\text{或者验证 } A^T A = \frac{1}{19} \begin{pmatrix} 15 & 10 & 6 \\ 6 & -15 & 10 \\ -10 & 6 & 16 \end{pmatrix} \frac{1}{19} \begin{pmatrix} 15 & 6 & -10 \\ 10 & -15 & 6 \\ 6 & 10 & 16 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

$$\text{又或者验证 } AA^T = \frac{1}{19} \begin{pmatrix} 15 & 6 & -10 \\ 10 & -15 & 6 \\ 6 & 10 & 16 \end{pmatrix} \frac{1}{19} \begin{pmatrix} 15 & 10 & 6 \\ 6 & -15 & 10 \\ -10 & 6 & 16 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I.$$

因为  $A$  是正交矩阵, 所以按定义  $y = Ax$  是正交变换.

(2)  $x = (3, 2, 0)^T$ , 求  $y$ ; 设  $y = (5, 2, 0)^T$ , 求  $x$ .

$$\text{若 } x = [3, 2, 0]^T, \text{ 则 } y = Ax = \frac{1}{19} \begin{pmatrix} 15 & 6 & -10 \\ 10 & -15 & 6 \\ 6 & 10 & 16 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$$

$$\text{若 } y = [5, 2, 0]^T, \text{ 则 } x = A^{-1}y = A^T y = \frac{1}{19} \begin{pmatrix} 15 & 10 & 6 \\ 6 & -15 & 10 \\ -10 & 6 & 16 \end{pmatrix} \begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ -2 \end{pmatrix}$$

**4.13** 4.13 如果  $\beta$  与  $\alpha_1, \alpha_2, \dots, \alpha_r$  都正交, 证明  $\beta$  与  $\alpha_1, \alpha_2, \dots, \alpha_r$  的任意线性组合也正交.

$$[\beta, k_1 \alpha_1 + k_2 \alpha_2 + \dots + k_r \alpha_r] = k_1 [\beta, \alpha_1] + k_2 [\beta, \alpha_2] + \dots + k_r [\beta, \alpha_r] = k_1 \cdot 0 + k_2 \cdot 0 + \dots + k_r \cdot 0 = 0.$$

**4.14** 4.14 设  $\alpha_1, \alpha_2, \dots, \alpha_n$  是一组  $n$  维线性无关向量,  $\beta$  是一个  $n$  维向量, 且  $[\beta, \alpha_i] = 0$  ( $i = 1, 2, \dots, n$ ), 证明  $\beta = 0$ .

因为  $\alpha_1, \alpha_2, \dots, \alpha_n$  线性无关, 而  $\alpha_1, \alpha_2, \dots, \alpha_n, \beta$  线性相关 ( $n+1$  个  $n$  维向量必然线性相关), 所以  $\beta$  可由  $\alpha_1, \alpha_2, \dots, \alpha_n$  线性表示:

$$\beta = k_1 \alpha_1 + k_2 \alpha_2 + \dots + k_n \alpha_n.$$

但  $\beta$  与  $\alpha_1, \alpha_2, \dots, \alpha_n$  都正交, 于是利用上题结论  $\beta$  与  $\alpha_1, \alpha_2, \dots, \alpha_n$  的任意线性组合也正交, 因此  $\beta$  与自己正交. 与自己正交的向量只有零向量.

**4.15** 4.15 若  $A$  是奇数阶正交矩阵,  $|A| = 1$ , 试证  $|A - I| = 0$ .

$$|A-E| = |AE - AA^T| = |A(E - A^T)| = |A| |(E - A^T)| = |A| |(E - A)^T| = |A| |E - A| = |A| \cdot (-1)^n |A - E| = |A| \cdot [-|A - E|]$$

现  $n$  是奇数,  $|A|=1$ , 故  $|A-E| = -|E-A|$ , 于是  $|A-E|=0$ .

**4.16** 4.16 已知矩阵  $A$ , 试求一个正交矩阵  $P$ , 使  $P^{-1}AP = A$

$$(1) A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad (2) A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$$

$$(1) \text{解特征方程 } |\lambda E - A| = \begin{vmatrix} \lambda-1 & -1 & -1 \\ -1 & \lambda-1 & -1 \\ -1 & -1 & \lambda-1 \end{vmatrix} \begin{matrix} r_2-r_2 \\ r_3-r_2 \\ r_3-r_2 \end{matrix} = \begin{vmatrix} \lambda-1 & -1 & -1 \\ -\lambda & \lambda & 0 \\ -\lambda & 0 & \lambda \end{vmatrix} \begin{matrix} \frac{1}{\lambda} r_2 \\ \frac{1}{\lambda} r_3 \\ r_1+r_2+r_3 \end{matrix} = \lambda^2 \begin{vmatrix} \lambda-3 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = \lambda^2 (\lambda-3)$$

得特征值  $\lambda_1 = \lambda_2 = 0, \lambda_3 = 3$ .

$$\text{对于 } \lambda_1 = \lambda_2 = 0, \lambda E - A = \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{得特征向量 } k_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + k_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} (k_1, k_2 \text{ 不全为 } 0). \text{ 选}$$

$$\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

$$\text{将 } \alpha_1, \alpha_2 \text{ 正交化: } \beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \beta_2 = \alpha_2 - \frac{[\alpha_2, \beta_1]}{[\beta_1, \beta_1]} \beta_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}.$$

$$\text{对于 } \lambda_3 = 3, \lambda E - A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \text{得特征向量 } k_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} (k_3 \neq 0). \text{ 选 } \beta_3 = \alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

$$\text{将 } \beta_1, \beta_2, \beta_3 \text{ 单位化: } \gamma_1 = \frac{1}{\|\beta_1\|} \beta_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \gamma_2 = \frac{1}{\|\beta_2\|} \beta_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}, \gamma_3 = \frac{1}{\|\beta_3\|} \beta_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\text{令 } P = (\gamma_1, \gamma_2, \gamma_3) = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}, \text{则 } P \text{ 是正交的, 且 } P^{-1}AP = A = \begin{pmatrix} 0 & & \\ & 0 & \\ & & 3 \end{pmatrix}.$$

**4.17** \*4.17 设三阶实对称阵  $A$  的特征值为  $\lambda_1 = -1, \lambda_2 = \lambda_3 = 1$ , 对应于  $\lambda_1$  的特征向量为  $\alpha_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ , 求对应于  $\lambda_2 = \lambda_3 = 1$  的特

征向量及  $A$ .

$$\text{设 } x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ 为对应于 } \lambda_2 = \lambda_3 = 1 \text{ 的特征向量, 则 } x \text{ 与 } \alpha_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \text{ 正交, 满足}$$

$$x_2 + x_3 = 0$$

解得  $\mathbf{x} = k_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + k_3 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$  ( $k_2, k_3$  不全为零). 可选  $\mathbf{a}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{a}_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ . 注意到  $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$  是两两正交的, 将之单位化后构成矩阵

$$\mathbf{P} = \left( \begin{array}{ccc} \frac{\mathbf{a}_1}{\|\mathbf{a}_1\|} & \frac{\mathbf{a}_2}{\|\mathbf{a}_2\|} & \frac{\mathbf{a}_3}{\|\mathbf{a}_3\|} \end{array} \right) = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix},$$

则  $\mathbf{P}$  是正交的, 且

$$\mathbf{P}^{-1} \mathbf{A} \mathbf{P} = \mathbf{A} = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}.$$

于是

$$\mathbf{A} = \mathbf{P} \mathbf{A} \mathbf{P}^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

**4.18** 4.18 求下列二次型的矩阵和秩.

(1)  $f(x_1, x_2, x_3) = 3x_1x_2 - 6x_1x_3 - x_2x_3 - 3x_3^2$

(2)  $f(x_1, x_2, x_3, x_4) = -5x_1^2 + 2x_3^2 + x_2x_3 - 5x_2x_4 - 10x_3x_4$

$$(1) \mathbf{A} = \begin{pmatrix} 0 & \frac{3}{2} & -3 \\ \frac{3}{2} & 0 & -\frac{1}{2} \\ -3 & -\frac{1}{2} & -3 \end{pmatrix} \xrightarrow[2c_2]{2r_2} \begin{pmatrix} 0 & 3 & -3 \\ 3 & 0 & -1 \\ -3 & -1 & -3 \end{pmatrix} \xrightarrow[\frac{1}{3}r_1]{\frac{1}{3}r_2} \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ -1 & -1 & -3 \end{pmatrix} \xrightarrow[r_3+r_1+r_2]{r_3-r_1+r_2} \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & -5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, R(\mathbf{A}) = 3$$

$$(2) \mathbf{A} = \begin{pmatrix} -5 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{5}{2} \\ 0 & \frac{1}{2} & 2 & -5 \\ 0 & -\frac{5}{2} & -5 & 0 \end{pmatrix} \xrightarrow[2c_2]{2r_2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -5 \\ 0 & 1 & 2 & -5 \\ 0 & -5 & -5 & 0 \end{pmatrix} \xrightarrow[\frac{1}{5}r_4]{\frac{1}{5}r_2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 2 & -1 \\ 0 & -1 & -1 & 0 \end{pmatrix} \xrightarrow[\frac{1}{5}c_4]{r_3-r_2+r_4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, R(\mathbf{A}) = 3$$

**4.19** 4.19 (1) 设二次型  $f$  的矩阵为  $\begin{pmatrix} 5 & 0.25 \\ 0.25 & -1 \end{pmatrix}$ , 求  $f(20, 10)$

(2) 设  $\mathbf{a} = [6, 4, 2, 0]^T, \mathbf{b} = [1, -3, -5, 7]^T$ , 二次型  $f$  的矩阵为  $\mathbf{a}\mathbf{a}^T$ , 求  $f(\mathbf{b})$ .

$$(1) f(x_1, x_2) = (x_1, x_2) \begin{pmatrix} 5 & 0.25 \\ 0.25 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} f(20, 10) = (20, 10) \begin{pmatrix} 5 & 0.25 \\ 0.25 & -1 \end{pmatrix} \begin{pmatrix} 20 \\ 10 \end{pmatrix} = 2000$$

$$(2) \mathbf{a}\mathbf{a}^T = \begin{pmatrix} 6 \\ 4 \\ 2 \\ 0 \end{pmatrix} (6, 4, 2, 0) = \begin{pmatrix} 36 & 24 & 12 & 0 \\ 24 & 16 & 8 & 0 \\ 12 & 8 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, f(\mathbf{x}) = \mathbf{x}^T \mathbf{a}\mathbf{a}^T \mathbf{x},$$

$$f(1, -3, -5, 7) = (1, -3, -5, 7) \begin{pmatrix} 36 & 24 & 12 & 0 \\ 24 & 16 & 8 & 0 \\ 12 & 8 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -3 \\ -5 \\ 7 \end{pmatrix} = 256$$

**4.20** 4.20 用正交变换化下列二次型为标准形

$$(1) f = 6x_1^2 + 24x_1x_2 - x_2^2$$

$$(2) f = 3x_1^2 + x_2^2 + x_3^2 - 6x_1x_2 - 6x_1x_3 - 2x_2x_3$$

(1) 二次型  $f$  的矩阵为  $A = \begin{pmatrix} 6 & 12 \\ 12 & -1 \end{pmatrix}$ . 现要对实对称矩阵  $A$  加以正交对角化, 即: 找出一个正交矩阵  $P$ , 使得

$P^{-1}AP = P^TAP = \Lambda$ , 其中  $\Lambda$  是对角阵(注意正交矩阵  $P$  满足  $P^{-1} = P^T$ ). 为此, 先求  $A$  的特征值.

解  $A$  的特征方程

$$|\lambda E - A| = \begin{vmatrix} \lambda - 6 & -12 \\ -12 & \lambda + 1 \end{vmatrix} = \lambda^2 - 5\lambda - 6 - 144 = 0,$$

得  $A$  的特征值  $\lambda_1 = -10, \lambda_2 = 15$ .

对于特征值  $\lambda_1 = -10$ , 解齐次方程组  $(\lambda E - A)x = 0$ . 方程组的系数矩阵

$$\lambda E - A = \begin{pmatrix} -16 & -12 \\ -12 & -9 \end{pmatrix} \rightarrow \begin{pmatrix} 4 & 3 \\ 0 & 0 \end{pmatrix}$$

得特征向量  $x = k_1 \begin{pmatrix} 3 \\ -4 \end{pmatrix}$  ( $k_1 \neq 0$ ). 取  $\alpha_1 = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$ .

对于特征值  $\lambda_2 = 15$ , 相应的齐次方程组的系数矩阵

$$\lambda E - A = \begin{pmatrix} 9 & -12 \\ -12 & 16 \end{pmatrix} \rightarrow \begin{pmatrix} -3 & 4 \\ 0 & 0 \end{pmatrix}$$

得特征向量  $x = k_2 \begin{pmatrix} 4 \\ 3 \end{pmatrix}$  ( $k_2 \neq 0$ ). 取  $\alpha_2 = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ .

$\alpha_1, \alpha_2$  是正交的, 将之单位化后构造正交矩阵

$$P = \begin{pmatrix} \frac{\alpha_1}{\|\alpha_1\|} & \frac{\alpha_2}{\|\alpha_2\|} \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{pmatrix}.$$

作正交变换  $x = Py$ , 即

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

则二次型

$$f = 6x_1^2 + 24x_1x_2 - x_2^2$$

化为标准形

$$f = \lambda_1 y_1^2 + \lambda_2 y_2^2 = -10y_1^2 + 15y_2^2.$$

(2) 二次型  $f$  的矩阵为  $A = \begin{pmatrix} 3 & -3 & -3 \\ -3 & 1 & -1 \\ -3 & -1 & 1 \end{pmatrix}$ . 解特征方程

$$|\lambda E - A| = \begin{vmatrix} \lambda - 3 & 3 & 3 \\ 3 & \lambda - 1 & 1 \\ 3 & 1 & \lambda - 1 \end{vmatrix} = 0,$$

得  $A$  的特征值  $\lambda_1 = -3, \lambda_2 = 2, \lambda_3 = 6$ .

对于特征值  $\lambda_1 = -3$ ,  $\lambda E - A = \begin{pmatrix} -6 & 3 & 3 \\ 3 & -4 & 1 \\ 3 & 1 & -4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$ , 取特征向量  $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ .

$$\text{对于特征值 } \lambda_2 = 2, \lambda \mathbf{E} - \mathbf{A} = \begin{pmatrix} -1 & 3 & 3 \\ 3 & 1 & 1 \\ 3 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \text{ 取特征向量 } \mathbf{a}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

$$\text{对于特征值 } \lambda_3 = 6, \lambda \mathbf{E} - \mathbf{A} = \begin{pmatrix} 3 & 3 & 3 \\ 3 & 5 & 1 \\ 3 & 1 & 5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}. \text{ 取特征向量 } \mathbf{a}_3 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}.$$

$\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$  是正交的. 令

$$\mathbf{P} = \left( \frac{\mathbf{a}_1}{\|\mathbf{a}_1\|}, \frac{\mathbf{a}_2}{\|\mathbf{a}_2\|}, \frac{\mathbf{a}_3}{\|\mathbf{a}_3\|} \right) = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix},$$

则  $\mathbf{P}$  是正交的. 作正交变换  $\mathbf{x} = \mathbf{P}\mathbf{y}$ , 则给出的二次型化为标准形

$$f = -3y_1^2 + 2y_2^2 + 6y_3^2.$$

**4.21** 4.21 设  $\mathbf{A} = \begin{pmatrix} 1 & -2 & -4 \\ -2 & 4 & -2 \\ -4 & -2 & 1 \end{pmatrix}$ , 试用正交变换化二次型  $f = \mathbf{x}^T \mathbf{A} \mathbf{x}$  为标准形.

解特征方程

$$|\lambda \mathbf{E} - \mathbf{A}| = \begin{vmatrix} \lambda - 1 & 2 & 4 \\ 2 & \lambda - 4 & 2 \\ 4 & 2 & \lambda - 1 \end{vmatrix} = 0,$$

得特征值  $\lambda_1 = -4, \lambda_2 = \lambda_3 = 5$ .

$$\text{对于特征值 } \lambda_1 = -4, \lambda \mathbf{E} - \mathbf{A} = \begin{pmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix}. \text{ 取特征向量 } \mathbf{a}_1 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}.$$

$$\text{对于特征值 } \lambda_2 = \lambda_3 = 5, \lambda \mathbf{E} - \mathbf{A} = \begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \text{ 取特征向量 } \mathbf{a}_2 = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \mathbf{a}_3 = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}.$$

$$\mathbf{a}_2, \mathbf{a}_3 \text{ 不是正交的, 将之正交化: } \boldsymbol{\beta}_2 = \mathbf{a}_2 = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \boldsymbol{\beta}_3 = \mathbf{a}_3 - \frac{[\mathbf{a}_3, \boldsymbol{\beta}_2]}{[\boldsymbol{\beta}_2, \boldsymbol{\beta}_2]} \boldsymbol{\beta}_2 = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} - \frac{4}{5} \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} -4 \\ -2 \\ 5 \end{pmatrix}.$$

$$\text{将 } \boldsymbol{\beta}_1 = \mathbf{a}_1, \boldsymbol{\beta}_2, \boldsymbol{\beta}_3 \text{ 单位化后构成 } \mathbf{P} = \left( \frac{\boldsymbol{\beta}_1}{\|\boldsymbol{\beta}_1\|}, \frac{\boldsymbol{\beta}_2}{\|\boldsymbol{\beta}_2\|}, \frac{\boldsymbol{\beta}_3}{\|\boldsymbol{\beta}_3\|} \right) = \begin{pmatrix} \frac{2}{3} & \frac{1}{\sqrt{5}} & -\frac{4}{3\sqrt{5}} \\ \frac{1}{3} & -\frac{2}{\sqrt{5}} & -\frac{2}{3\sqrt{5}} \\ \frac{2}{3} & 0 & \frac{5}{3\sqrt{5}} \end{pmatrix}, \text{ 则 } \mathbf{P} \text{ 是正交的. 作正交变换 } \mathbf{x} = \mathbf{P}\mathbf{y}, \text{ 则二次型化}$$

为标准形

$$f = -4y_1^2 + 5y_2^2 + 5y_3^2.$$

† 上面对于特征值  $\lambda_2 = \lambda_3 = 5$ , 可取特征向量  $\mathbf{a}_2 = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}, \mathbf{a}_3 = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}$ . 这样  $\mathbf{a}_2, \mathbf{a}_3$  是正交的. 可令

$$\mathbf{P} = \left( \frac{\mathbf{a}_1}{\|\mathbf{a}_1\|}, \frac{\mathbf{a}_2}{\|\mathbf{a}_2\|}, \frac{\mathbf{a}_3}{\|\mathbf{a}_3\|} \right) = \frac{1}{3} \begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & -2 \\ 2 & -2 & -1 \end{pmatrix}.$$

这样, 得到不同的变换矩阵, 但标准形仍然同上, 因为构造  $\mathbf{P}$  所用的特征向量  $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$  仍然依次对应着特征值  $\lambda_1 = -4, \lambda_2 = \lambda_3 = 5$ .

**4.22** 4.22 用可逆线性变换将如下二次型化为标准形, 并求出正惯性指标:

$$(1) f = x_1x_2 + x_1x_3 + x_2x_3$$

$$(2) f = x_1^2 + 2x_2^2 + x_4^2 + 4x_1x_2 + 4x_1x_3 + 2x_1x_4 + 2x_2x_3 + 2x_2x_4 + 2x_3x_4$$

(1) 二次型  $f$  的矩阵为 
$$A = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}.$$

解特征方程

$$\begin{aligned} |\lambda I - A| &= \begin{vmatrix} \lambda & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \lambda & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \lambda \end{vmatrix} \stackrel{\substack{r_2-r_1 \\ r_3-r_1}}{=} \begin{vmatrix} \lambda & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2}-\lambda & \lambda+\frac{1}{2} & 0 \\ -\frac{1}{2}-\lambda & 0 & \lambda+\frac{1}{2} \end{vmatrix}, \\ &= \left(\lambda + \frac{1}{2}\right)^2 \begin{vmatrix} \lambda & -\frac{1}{2} & -\frac{1}{2} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} \stackrel{r_1 + \frac{1}{2}r_2 + \frac{1}{2}r_3}{=} \left(\lambda + \frac{1}{2}\right)^2 \begin{vmatrix} \lambda-1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = \left(\lambda + \frac{1}{2}\right)^2 (\lambda-1) = 0 \end{aligned}$$

得特征值  $\lambda_1 = \lambda_2 = -\frac{1}{2}, \lambda_3 = 1$ .

对于特征值  $\lambda_1 = \lambda_2 = -\frac{1}{2}$ ,  $\lambda E - A = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ . 取特征向量  $\alpha_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ .

$\alpha_2, \alpha_3$  不是正交的, 将之正交化:  $\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \beta_2 = \alpha_2 - \frac{[\alpha_2, \beta_1]}{[\beta_1, \beta_1]} \beta_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$ .

对于特征值  $\lambda_3 = 1, \lambda E - A = \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$ . 取特征向量  $\beta_3 = \alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ .

令  $P = \left( \frac{\beta_1}{\|\beta_1\|}, \frac{\beta_2}{\|\beta_2\|}, \frac{\beta_3}{\|\beta_3\|} \right) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}$ . 则  $P$  是正交的. 作正交变换  $x = Py$ , 则二次型化为标准形

$$f = -\frac{1}{2}y_1^2 - \frac{1}{2}y_2^2 + y_3^2.$$

正惯性指标是 1.

**解法二** 本题的变换矩阵不要求是正交的, 可用配方法.

取

$$\begin{cases} x_1 = y_1 + y_2 \\ x_2 = y_1 - y_2 \\ x_3 = y_3 \end{cases}$$

则



$$\begin{aligned}
 f &= x_1x_2 + x_1x_3 + x_2x_3 \\
 &= (y_1 + y_2)(y_1 - y_2) + (y_1 + y_2)y_3 + (y_1 - y_2)y_3 \\
 &= y_1^2 - y_2^2 + 2y_1y_3 \\
 &= (y_1 + y_3)^2 - y_2^2 - y_3^2 \\
 &= z_1^2 - z_2^2 - z_3^2,
 \end{aligned}$$

其中

$$\begin{cases} z_1 = y_1 + y_3 \\ z_2 = y_2 \\ z_3 = y_3 \end{cases}.$$

而

$$\begin{cases} y_1 = z_1 - z_3 \\ y_2 = z_2 \\ y_3 = z_3 \end{cases}, \quad \begin{cases} x_1 = (z_1 - z_3) + z_2 = z_1 + z_2 - z_3 \\ x_2 = (z_1 - z_3) - z_2 = z_1 - z_2 - z_3 \\ x_3 = z_3 \end{cases}.$$

可见作可逆线性变换

$$\mathbf{x} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{z},$$

则给出的二次型就化为标准形

$$f = z_1^2 - z_2^2 - z_3^2.$$

(2) 二次型  $f$  的矩阵  $A = \begin{pmatrix} 1 & 2 & 2 & 1 \\ 2 & 2 & 1 & 1 \\ 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ .

特征方程

$$|\lambda E - A| = \begin{vmatrix} \lambda-1 & -2 & -2 & -1 \\ -2 & \lambda-2 & -1 & -1 \\ -2 & -1 & \lambda & -1 \\ -1 & -1 & -1 & \lambda-1 \end{vmatrix} \stackrel{r_4 - r_1 + r_2 - r_3}{=} \begin{vmatrix} \lambda-1 & -2 & -2 & -1 \\ -2 & \lambda-2 & -1 & -1 \\ -2 & -1 & \lambda & -1 \\ -\lambda & \lambda & -\lambda & \lambda \end{vmatrix} = \lambda \begin{vmatrix} \lambda-1 & -2 & -2 & -1 \\ -2 & \lambda-2 & -1 & -1 \\ -2 & -1 & \lambda & -1 \\ -1 & 1 & -1 & 1 \end{vmatrix} = \lambda \begin{vmatrix} \lambda-2 & -1 & -3 \\ -3 & \lambda-1 & -2 \\ -3 & 0 & \lambda-1 \end{vmatrix} = 0,$$

的根不容易求. 下面的结果是用 matlab 得到的.

特征值  $\lambda_1 = -\frac{1640}{987}, \lambda_2 = 0, \lambda_3 = \frac{301}{650}, \lambda_4 = \frac{6337}{1219}$ .

正交变换矩阵  $P = \begin{pmatrix} \frac{646}{995} & \frac{1}{2} & \frac{178}{4561} & \frac{219}{383} \\ \frac{583}{3205} & \frac{1}{2} & \frac{634}{1067} & \frac{377}{625} \\ \frac{633}{865} & \frac{1}{2} & \frac{285}{1307} & \frac{610}{1493} \\ -\frac{371}{3734} & \frac{1}{2} & \frac{1715}{2218} & \frac{347}{920} \end{pmatrix}$ .

则作正交变换  $\mathbf{x} = P\mathbf{y}$ , 二次型就化为标准形  $f = -\frac{1640}{987}y_1^2 + 0 \cdot y_2^2 + \frac{301}{650}y_3^2 + \frac{6337}{1219}y_4^2$ . 显见正惯性指标  $p = 2$ .

**解法二** 下面用配方法计算

$$\begin{aligned}
 f &= x_1^2 + 2x_2^2 + x_4^2 + 4x_1x_2 + 4x_1x_3 + 2x_1x_4 + 2x_2x_3 + 2x_2x_4 + 2x_3x_4 \\
 &= (x_1^2 + 4x_1x_2 + 4x_1x_3 + 2x_1x_4) + 2x_2^2 + x_4^2 + 2x_2x_3 + 2x_2x_4 + 2x_3x_4 \\
 &= (x_1 + 2x_2 + 2x_3 + x_4)^2 - (2x_2 + 2x_3 + x_4)^2 + 2x_2^2 + x_4^2 + 2x_2x_3 + 2x_2x_4 + 2x_3x_4 \\
 &= y_1^2 - 2(x_2^2 + 3x_2x_3 + x_2x_4) - 4x_3^2 - 2x_3x_4 \\
 &= y_1^2 - 2\left(x_2 + \frac{3}{2}x_3 + \frac{1}{2}x_4\right)^2 + 2\left(\frac{3}{2}x_3 + \frac{1}{2}x_4\right)^2 - 4x_3^2 - 2x_3x_4 \\
 &= y_1^2 - 2y_2^2 + \frac{1}{2}(x_3^2 + 2x_3x_4 + x_4^2) \\
 &= y_1^2 - 2y_2^2 + \frac{1}{2}(x_3 + x_4)^2 \\
 &= y_1^2 - 2y_2^2 + \frac{1}{2}y_3^2
 \end{aligned}$$

其中

$$\begin{cases} y_1 = x_1 + 2x_2 + 2x_3 + x_4 \\ y_2 = x_2 + \frac{3}{2}x_3 + \frac{1}{2}x_4 \\ y_3 = x_3 + x_4 \\ y_4 = x_4 \end{cases} \quad \text{或} \quad \mathbf{y} = \begin{pmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mathbf{x}.$$

可见作是可逆线性变换

$$\mathbf{x} = \begin{pmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} \mathbf{y} = \begin{pmatrix} 1 & -2 & 1 & -1 \\ 0 & 1 & -\frac{3}{2} & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mathbf{y}$$

则二次型化成标准形  $f = y_1^2 - 2y_2^2 + \frac{1}{2}y_3^2$ . 正惯性指标  $p = 3$ .

**4.23** 4.23 判别下列二次型的正定性:

(1)  $f = 2x_1^2 - 6x_2^2 - 4x_3^2 + 2x_1x_2 + 2x_1x_3$

(2)  $f = x_1^2 + 3x_2^2 + 9x_3^2 + 19x_4^2 - 2x_1x_2 + 4x_1x_3 + 2x_1x_4 - 6x_2x_4 - 12x_3x_4$

(1) 二次型的矩阵  $\mathbf{A} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & -6 & 0 \\ 1 & 0 & -4 \end{pmatrix}$  的各阶主子式依次为

$$\mathcal{A}_1 = 2 > 0, \quad \mathcal{A}_2 = \begin{vmatrix} 2 & 1 \\ 1 & -6 \end{vmatrix} = -14 < 0, \quad \mathcal{A}_3 = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -6 & 0 \\ 1 & 0 & -4 \end{vmatrix} = 58 > 0.$$

故二次型是负定的.

(2) 二次型的矩阵  $\mathbf{A} = \begin{pmatrix} 1 & -1 & 2 & 1 \\ -1 & 3 & 0 & -3 \\ 2 & 0 & 9 & -6 \\ 1 & -3 & -6 & 19 \end{pmatrix}$  的各阶主子式依次为

$$\mathcal{A}_1 = 1 > 0, \quad \mathcal{A}_2 = \begin{vmatrix} 1 & -1 \\ -1 & 3 \end{vmatrix} = 2 > 0, \quad \mathcal{A}_3 = \begin{vmatrix} 1 & -1 & 2 \\ -1 & 3 & 0 \\ 2 & 0 & 9 \end{vmatrix} = 6 > 0, \quad \mathcal{A}_4 = |\mathbf{A}| = 24 > 0.$$

故二次型是正定的.

**4.24** 4.24  $\lambda$  取何值时, 下列二次型是正定的:

(1)  $f = x_1^2 + x_2^2 + 5x_3^2 + 2\lambda x_1x_2 - 2x_1x_3 + 4x_2x_3$

(2)  $f = x_1^2 + 4x_2^2 + x_3^2 + 2\lambda x_1x_2 + 10x_1x_3 + 6x_2x_3$

(1) 二次型是正定的, 当且仅当它的矩阵  $\mathbf{A} = \begin{pmatrix} 1 & \lambda & -1 \\ \lambda & 1 & 2 \\ -1 & 2 & 5 \end{pmatrix}$  的各阶主子式都是正的:

$$A_1 = 1 > 0, \quad A_2 = \begin{vmatrix} 1 & \lambda \\ \lambda & 1 \end{vmatrix} = 1 - \lambda^2 > 0, \quad A_3 = \begin{vmatrix} 1 & \lambda & -1 \\ \lambda & 1 & 2 \\ -1 & 2 & 5 \end{vmatrix} = -5\lambda^2 - 4\lambda > 0,$$

$$\text{即} \begin{cases} 1 - \lambda^2 > 0 \\ -5\lambda^2 - 4\lambda > 0 \end{cases} \text{解得 } -\frac{4}{5} < \lambda < 0.$$

(2) 二次型是正定的, 当且仅当它的矩阵  $A = \begin{pmatrix} 1 & \lambda & 5 \\ \lambda & 4 & 3 \\ 5 & 3 & 1 \end{pmatrix}$  的各阶主子式都是正的:

$$A_1 = 1 > 0, \quad A_2 = \begin{vmatrix} 1 & \lambda \\ \lambda & 4 \end{vmatrix} = 4 - \lambda^2 > 0, \quad A_3 = \begin{vmatrix} 1 & \lambda & 5 \\ \lambda & 4 & 3 \\ 5 & 3 & 1 \end{vmatrix} = -\lambda^2 + 30\lambda - 105 > 0,$$

$$\text{即} \begin{cases} 4 - \lambda^2 > 0 \\ -\lambda^2 + 30\lambda - 105 > 0 \end{cases}, \text{ 或 } \begin{cases} -2 < \lambda < 2 \\ -\lambda^2 + 30\lambda - 105 > 0 \end{cases}. \text{ 当 } \lambda < 2 \text{ 时, } -\lambda^2 + 30\lambda - 105 < 0. \text{ 这说明无论 } \lambda \text{ 取何值, } f \text{ 皆非正定.}$$

**4.25** 4.25 验证二次型  $f(x_1, x_2, x_3) = 4x_1^2 + 5x_2^2 + 4x_3^2 + 2x_1x_2 + 2x_2x_3$  是正定二次型; 用正交变换把椭球面方程

$f(x_1, x_2, x_3) = 24$  标准化, 并求出它的三个轴长.

二次型的矩阵  $A = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 5 & 1 \\ 0 & 1 & 4 \end{pmatrix}$  的各阶主子式都是正的:

$$A_1 = 4 > 0, \quad A_2 = \begin{vmatrix} 4 & 1 \\ 1 & 5 \end{vmatrix} = 19 > 0, \quad A_3 = \begin{vmatrix} 4 & 1 & 0 \\ 1 & 5 & 1 \\ 0 & 1 & 4 \end{vmatrix} = 72 > 0.$$

故二次型是正定的.

$A$  的特征值  $\lambda_1 = 3, \lambda_2 = 4, \lambda_3 = 6$  可通过解特征方程  $\lambda E - A = \begin{vmatrix} \lambda - 4 & -1 & 0 \\ -1 & \lambda - 5 & -1 \\ 0 & -1 & \lambda - 4 \end{vmatrix} = 0$  得到, 相应的特征向量通过求解齐次

$$\text{方程组 } (\lambda E - A)\mathbf{x} = 0 \text{ 分别可取为 } \mathbf{a}_1 = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}, \mathbf{a}_2 = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \mathbf{a}_3 = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{\sqrt{2}}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}.$$

$\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$  构成标准正交向量组, 由它们构成正交矩阵

$$P = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & 0 & \frac{\sqrt{2}}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix} = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{2} & -\sqrt{3} & 1 \\ -\sqrt{2} & 0 & 2 \\ \sqrt{2} & \sqrt{3} & 1 \end{pmatrix}.$$

作正交变换  $\mathbf{x} = P\mathbf{y}$ , 则有  $f = 3y_1^2 + 4y_2^2 + 6y_3^2$ . 于是在新的坐标系  $Oy_1y_2y_3$  下椭球面  $f(x_1, x_2, x_3) = 24$  的方程为

$$f = 3y_1^2 + 4y_2^2 + 6y_3^2 = 24, \text{ 或标准化为 } \frac{y_1^2}{(2\sqrt{2})^2} + \frac{y_2^2}{(\sqrt{6})^2} + \frac{y_3^2}{2^2} = 1. \text{ 椭球面的三个半轴的长度依次为 } a = 2\sqrt{2}, b = \sqrt{6}, c = 2.$$

**4.26** 4.26 已知二次型  $f(x_1, x_2, x_3) = 2x_1^2 + 3x_2^2 + 3x_3^2 + 2ax_2x_3$  ( $a > 0$ ), 通过正交变换化成标准形  $f = y_1^2 + 2y_2^2 + 5y_3^2$ , 求参数  $a$  及所用正交变换矩阵.

$$\text{给出的二次型的矩阵是 } A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & a \\ 0 & a & 3 \end{pmatrix}.$$

因为二次型经正交变换后所得标准形的系数是二次型的矩阵的特征值, 所以  $A$  的特征值  $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 5$ . 由  $|A| = \lambda_1 \lambda_2 \lambda_3$ , 或  $2(9 - a^2) = 1 \cdot 2 \cdot 5$  可确定  $a = \pm 2$ .

下面先对  $a = 2$  进行讨论.

$$\text{对于 } \lambda_1 = 1, \lambda E - A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \text{ 得特征向量 } k_1 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} (k_1 \neq 0).$$

$$\text{对于 } \lambda_2 = 2, \lambda E - A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & -2 \\ 0 & -2 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \text{ 得特征向量 } k_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} (k_2 \neq 0).$$

$$\text{对于 } \lambda_3 = 5, \lambda E - A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}. \text{ 得特征向量 } k_3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} (k_3 \neq 0).$$

这三组相互正交的特征向量的单位向量分别是  $\alpha_1 = \pm \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}, \alpha_2 = \pm \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \alpha_3 = \pm \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ . 以之构成矩阵  $P = (\alpha_1, \alpha_2, \alpha_3)$ , 则

$P$  是所求的正交变换矩阵.  $P$  不是唯一的, 可取

$$P = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

对  $a = -2$ , 类似可算得三个两两正交的单位特征向量:  $\alpha_1 = \pm \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \alpha_2 = \pm \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \alpha_3 = \pm \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$ . 可取

$$P = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

**4.27** 4.27 证明: 二次型  $f = \mathbf{x}^T A \mathbf{x}$  在  $\|\mathbf{x}\| = 1$  时的最大值为实对称矩阵  $A$  的最大特征值.

有正交变换  $\mathbf{x} = P\mathbf{y}$ , 使二次型化成标准形  $f = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2$ , 其中  $\lambda_1, \lambda_2, \cdots, \lambda_n$  是  $A$  的特征值(按由小到大的次序排列).

正交变换不改变向量的模, 故  $\|\mathbf{y}\| = \|\mathbf{x}\| = 1$ . 于是在  $\|\mathbf{x}\| = 1$  时,

$$f = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2 \leq \lambda_n (y_1^2 + y_2^2 + \cdots + y_n^2) \leq \lambda_n \|\mathbf{y}\|^2 = \lambda_n.$$

取  $\mathbf{y} = \boldsymbol{\varepsilon}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$ , 当  $\mathbf{x} = P\boldsymbol{\varepsilon}_n$  时,  $f(\mathbf{x}) = \lambda_n$ , 这是  $f$  的最大值.

**4.28** \*4.28 设  $A$  为正定矩阵, 试证  $A^2, A^*$  也均为正定矩阵.

当  $A$  是对称矩阵时,  $A^2, A^*$  也都是对称矩阵:

$$(A^2)^T = (A^T)^2 = A^2,$$

$$(A^*)^T = (|A|A^{-1})^T = |A|(A^T)^{-1} = |A|A^{-1} = A^*.$$

$A$  的所有特征值  $\lambda_i (i = 1, 2, \cdots, n)$  都大于零. 从而  $|A| > 0$ .

$A^2$  的所有特征值  $\lambda_i^2 (i = 1, 2, \cdots, n)$  也都大于零, 所以  $A^2$  是正定的.

$A^* = |A|A^{-1}$  的所有特征值  $|A|\lambda_i^{-1}$  ( $i=1,2,\dots,n$ ) 也都大于零, 所以  $A^*$  是正定的.

**4.29** 4.29 设  $A, B$  都是  $n$  阶正定矩阵, 证明  $A+B$  也是正定矩阵.

$A, B$  都是正定矩阵, 所以二次型  $x^T Ax, x^T Bx$  都是正定的, 即

$$x^T Ax > 0 (x \neq 0),$$

$$x^T Bx > 0 (x \neq 0).$$

于是

$$x^T (A+B)x = x^T Ax + x^T Bx > 0 (x \neq 0),$$

这说明  $A+B$  也是正定矩阵.

**4.30** 4.30 设  $A$  是  $n$  阶实对称矩阵, 证明存在实数  $c$ , 对一切  $x \in \mathbb{R}^n$ , 有  $|x^T Ax| \leq cx^T x$ .

存在正交变换  $x = Py$ , 使二次型

$$x^T Ax = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \dots + \lambda_n y_n^2.$$

设  $\tilde{\lambda}$  是  $A$  的绝对值最大的特征值, 则  $\forall x \in \mathbb{R}^n$

$$\begin{aligned} |x^T Ax| &= |\lambda_1 y_1^2 + \lambda_2 y_2^2 + \dots + \lambda_n y_n^2| \\ &\leq |\lambda_1| y_1^2 + |\lambda_2| y_2^2 + \dots + |\lambda_n| y_n^2 \\ &\leq |\tilde{\lambda}| (y_1^2 + y_2^2 + \dots + y_n^2) \\ &= |\tilde{\lambda}| y^T y. \end{aligned}$$

由于  $x = Py$  是正交变换, 所以  $y^T y = x^T x$ . 于是  $\forall x \in \mathbb{R}^n, |x^T Ax| \leq |\tilde{\lambda}| x^T x$ . 取  $c = |\tilde{\lambda}|$  即得证.

**4.31** 4.31 设  $A$  是  $n$  阶正定矩阵,  $I$  是  $n$  阶单位矩阵, 证明:  $\det(I+A) > 1$ .

由  $A$  的正定性知  $A$  的特征值  $\lambda_i > 0$  ( $i=1,2,\dots,n$ ). 于是  $I+A$  的特征值  $1+\lambda_i > 1$  ( $i=1,2,\dots,n$ ), 从而  $|I+A| = (1+\lambda_1) \cdot (1+\lambda_2) \cdot \dots \cdot (1+\lambda_n) > 1$ .

**4.32** 4.32 设  $A$  为  $n$  阶实对称矩阵, 且  $A^3 - 3A^2 + 5A - 3I = 0$ . 证明:  $A$  正定.

$A$  的特征值  $\lambda$  满足  $\lambda^3 - 3\lambda^2 + 5\lambda - 3 = (\lambda-1)(\lambda^2 - 2\lambda + 3) = 0$ , 或  $\lambda = 1, 1 \pm i\sqrt{2}$ . 但实对称矩阵的特征值都是实数, 故  $A$  的特征值  $\lambda_1 = \lambda_2 = \lambda_3 = 1$ . 从而  $A$  是正定的.

**4.33** 4.33 填空题:

(1) 设向量  $\alpha = [-1, x, 2]^T$  与  $\beta = [3x, x, -5]^T$  正交, 则  $x = \underline{-2, 5}$ .

(2) 向量  $\alpha = [-1, 4, 0, 2]^T$  与  $\beta = [2, -2, 1, 3]^T$  的距离和内积分别为  $\underline{-4}$  和  $\underline{5\sqrt{2}}$ .

(3) 向量  $\alpha = [1, 2, 2, 3]^T$  与  $\beta = [3, 1, 5, 1]^T$  的夹角为  $\underline{\frac{\pi}{4}}$ .

(4) 向量组  $\alpha_1 = [0, 0, 1]^T, \alpha_2 = [0, 1, 1]^T, \alpha_3 = [1, 1, 1]^T$  经施密特方法正交规范化为  $\beta_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \beta_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \beta_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ .

(5) 若矩阵  $A = \begin{bmatrix} x & 0 & 0 \\ 0 & y & z \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$  是正交矩阵, 则  $x, y, z$  的值分别为  $x = \pm 1, y = -z = \pm \frac{1}{\sqrt{2}}$ .

(1)  $-3x + x^2 - 10 = 0, x = -2, 5$

(2) 向量  $\alpha = [-1, 4, 0, 2]^T$  与  $\beta = [2, -2, 1, 3]^T$  内积为

$$[\alpha, \beta] = \begin{bmatrix} (-1) & \begin{pmatrix} 2 \\ -2 \\ 1 \\ 3 \end{pmatrix} \\ 4 & \\ 0 & \\ 2 & \end{bmatrix} = (-1) \cdot 2 + 4 \cdot (-2) + 0 \cdot 1 + 2 \cdot 3 = -4;$$

距离为

$$\|\alpha - \beta\| = \left\| \begin{pmatrix} -1 \\ 4 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \\ 1 \\ 3 \end{pmatrix} \right\| = \left\| \begin{pmatrix} -3 \\ 6 \\ -2 \\ -1 \end{pmatrix} \right\| = \sqrt{(-3)^2 + 6^2 + (-2)^2 + (-1)^2} = \sqrt{50} = 5\sqrt{2}.$$

(3) 向量  $\alpha = [1, 2, 2, 3]^T$  与  $\beta = [3, 1, 5, 1]^T$  的夹角为

$$\arccos \frac{[\alpha, \beta]}{\|\alpha\| \|\beta\|} = \arccos \frac{18}{\sqrt{36} \sqrt{18}} = \arccos \frac{1}{\sqrt{2}} = \frac{\pi}{4}.$$

(4) 向量组  $\alpha_1 = [0, 0, 1]^T$ ,  $\alpha_2 = [0, 1, 1]^T$ ,  $\alpha_3 = [1, 1, 1]^T$  经施密特方法正交规范化为

$$\begin{aligned} \beta_1 &= \alpha_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \\ \beta_2 &= \alpha_2 - \frac{[\alpha_2, \beta_1]}{[\beta_1, \beta_1]} \beta_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{1} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \\ \beta_3 &= \alpha_3 - \frac{[\alpha_3, \beta_1]}{[\beta_1, \beta_1]} \beta_1 - \frac{[\alpha_3, \beta_2]}{[\beta_2, \beta_2]} \beta_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{1} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{1} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \end{aligned}$$

得到的  $\beta_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ ,  $\beta_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ ,  $\beta_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$  已经是单位向量, 为所求的正交规范组.

$$(5) A = \begin{pmatrix} x & 0 & 0 \\ 0 & y & z \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \text{ 的行向量组是标准正交向量组, 满足 } \begin{cases} [\alpha_1, \alpha_1] = x^2 = 1 \\ [\alpha_2, \alpha_2] = y^2 + z^2 = 1 \\ [\alpha_2, \alpha_3] = y \cdot \frac{1}{\sqrt{2}} + z \cdot \frac{1}{\sqrt{2}} = 0 \end{cases}.$$

解得  $x = \pm 1, y = -z = \pm \frac{1}{\sqrt{2}}$

(6) 设三阶实对称方阵  $A$  的特征值为  $\lambda_1 = \lambda_2 = 1, \lambda_3 = 2$ , 则  $R(A - 2E) = \underline{2}$ .  $R(A - E) = \underline{1}$ .

**定理 3.10** 若  $\lambda_i$  是实对称矩阵  $A$  的  $k$  重特征值, 则存在  $k$  个属于  $\lambda_i$  的线性无关的特征向量.

$\lambda_3 = 2$  是 1 重特征值,  $A$  有 1 个属于  $\lambda_3 = 2$  的线性无关的特征向量, 即齐次方程组  $(A - 2E)x = 0$  的基础解系含 1 个解向量. 于是  $n - R(A - 2E) = 1$ . 现  $n = 3$ , 故  $R(A - 2E) = 2$ . 类似地,  $\lambda_1 = \lambda_2 = 1$  是 2 重特征值, 方程组  $(A - E)x = 0$  的基础解系含 2 个解向量. 于是  $n - R(A - E) = 2$ , 而  $R(A - E) = 1$ .

(7) 二次型  $f(x_1, x_2, x_3, x_4) = x_1^2 + 2x_2^2 + 3x_3^2 + 4x_1x_2 + 2x_2x_3$  的矩阵是  $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$ , 该二次型的秩为 3.

(8) 矩阵  $A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 2 & -1 \\ 4 & -1 & 3 \end{pmatrix}$  对应的二次型是  $f = x_1^2 + 2x_2^2 + 3x_3^2 + 4x_1x_2 + 8x_1x_3 - 2x_2x_3$ .

(9) 已知二次型  $f(x_1, x_2, x_3) = 5x_1^2 + 5x_2^2 + cx_3^2 - 2x_1x_2 + 6x_1x_3 - 6x_2x_3$  的秩为 2, 则参数  $c = \underline{3}$ .

$$A = \begin{pmatrix} 5 & -1 & 3 \\ -1 & 5 & -3 \\ 3 & -3 & c \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & c-3 \end{pmatrix}$$

(10) 设  $A$  是实对称可逆矩阵, 则将  $f = x^T Ax$  化为  $f = y^T A^{-1}y$  的线性变换为  $x = A^{-1}y$ .

**解法一**  $f = x^T Ax = x^T AA^{-1}Ax = x^T A^{-1}A^T Ax = (Ax)^T A^{-1}(Ax)$ , 令  $y = Ax$ , 即作线性变换  $x = A^{-1}y$ , 则  $f = x^T Ax$  化为  $f = y^T A^{-1}y$ .

**解法二**  $f = y^T A^{-1}y = y^T A^{-1}AA^{-1}y = y^T (A^T)^{-1}AA^{-1}y = y^T (A^{-1})^T AA^{-1}y = (A^{-1}y)^T A(A^{-1}y)$ . 作线性变换  $x = A^{-1}y$ , 则  $f = y^T A^{-1}y = x^T A^{-1}x$

(11) 设  $n$  阶实对称矩阵  $A$  的特征值分别为  $1, 2, \dots, n$ , 则当  $t > n$  时,  $tI - A$  为正定矩阵.

当  $t > 0$  时,  $tI - A$  的特征值  $t-1, t-2, \dots, t-n$  都大于零.

4.34 4.34 选择题

(1) 设  $\alpha = [-1, 1, 5, -3]^T, \beta = [-9, -2, 3, -5]^T$ , 则  $\alpha$  与  $\beta$  的距离为 (B).

- (A)81 (B)9 (C)7 (D)8

(2) 设  $n$  维向量  $\alpha$  和  $\beta$  的模分别是 6 和 9,  $\alpha$  与  $\beta$  的距离是 12, 则  $\alpha$  与  $\beta$  的夹角的余弦为 (A).

- (A)  $-\frac{1}{4}$  (B)  $\frac{1}{4}$  (C)  $\frac{1}{2}$  (D)  $-\frac{1}{2}$

书上 p87 例 4.6 的公式:  $\|\alpha - \beta\| = \sqrt{\|\alpha\|^2 + \|\beta\|^2 - 2\|\alpha\| \cdot \|\beta\| \cos \theta}$

$$12 = \sqrt{6^2 + 9^2 - 2 \cdot 6 \cdot 9 \cdot \cos \theta} \Rightarrow \cos \theta = -\frac{1}{4}$$

(3) 设  $A$  是  $n$  阶方阵, 则下列 4 个式子中表明  $A$  是正交矩阵的式子为 (C).

- (A)  $AA^{-1} = I$  (B)  $AA = I$  (C)  $A^T = A^{-1}$  (D)  $|A| = \pm 1$

(4) 设  $\alpha = [8, -4, 1]^T$ , 若  $\beta$  与  $\alpha$  是同方向的向量, 且  $\|\beta\| = 6, \|\alpha\| = 9$ , 则  $\beta =$  (D).

- (A)  $[16, -8, 2]^T$  (B)  $[-16, -8, 2]^T$  (C)  $[-\frac{16}{3}, -\frac{8}{3}, \frac{2}{3}]^T$  (D)  $[\frac{16}{3}, -\frac{8}{3}, \frac{2}{3}]^T$

$\beta$  等于  $\alpha$  方向上的单位向量的  $\|\beta\|$  倍:  $\beta = \|\beta\| \left( \frac{1}{\|\alpha\|} \alpha \right) = 6 \left( \frac{1}{9} [8, -4, 1]^T \right) = \left[ \frac{16}{3}, -\frac{8}{3}, \frac{2}{3} \right]^T$ .

(5) 设  $\alpha$  和  $\beta$  是  $n$  维向量, 则等式  $\|\alpha + \beta\|^2 = \|\alpha\|^2 + \|\beta\|^2$  成立的充分必要条件是 (C).

- (A)  $[\alpha, \beta] > 0$  (B)  $[\alpha, \beta] < 0$  (C)  $[\alpha, \beta] = 0$  (D)  $[\alpha, \beta] \neq 0$

(6) 设  $A, B$  均为  $n$  阶方阵,  $x = (\chi_1, \chi_2, \dots, \chi_n)^T$ , 且  $x^T Ax = x^T Bx$ , 当 (D) 时,  $A = B$

- (A) 秩( $A$ ) = 秩( $B$ ) (B)  $A^T = A$   
(C)  $B^T = B$  (D)  $A^T = A$  且  $B^T = B$

当  $A^T = A$  且  $B^T = B$  时,  $A, B$  是同一个二次型的矩阵, 而二次型的矩阵是唯一的, 故  $A = B$ .

(7) 下列矩阵为正定的是 (D)

- (A)  $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$  (B)  $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}$  (C)  $\begin{pmatrix} 1 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & -2 \end{pmatrix}$  (D)  $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 5 \end{pmatrix}$

(A)的矩阵不是正定的, 因其二阶主子式  $\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1 < 0$ .

(B)的矩阵不是正定的, 因其二阶主子式  $\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0$ .

(C)的矩阵不是正定的, 因其三阶主子式  $\begin{vmatrix} 1 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & -2 \end{vmatrix} = -2 \begin{vmatrix} 1 & -2 \\ -2 & 5 \end{vmatrix} = -2 \cdot 1 < 0$ .

(D)的矩阵不是正定的, 因其各阶主子式都大于零:

$$\Delta_1 = 2 > 0, \quad \Delta_2 = \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} = 2 > 0, \quad \Delta_3 = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 5 \end{vmatrix} = 2 \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = 2 \cdot 1 = 2 > 0.$$

(8) 实二次型  $f(x_1, x_2, \dots, x_n) = x^T A x$  为正定的充分必要条件是(B).

(A)  $|A| > 0$       (B) 存在  $n$  阶可逆矩阵  $C$ , 使  $A = C^T C$

(C) 负惯性指标为零      (D) 对某一  $x \neq 0$ , 有  $x^T A x > 0$

(A)只是必要条件.

矩阵  $A$  的正惯性指标  $p$  与负惯性指标  $\tilde{p}$  之和等于  $A$  的秩  $r$ :  $p + \tilde{p} = r$ . 而  $\tilde{p} = 0$  时  $p = r$ , 但这不保证正惯性指标为  $n$ . 故不选(C).

(D)中“对某一  $x \neq 0$ ”要改为“对任一  $x \neq 0$ ”才符合正定二次型的定义. 故不选(D).

根据定理 5.6 选(B).

**定理 4.6** 若  $A$  是  $n$  阶实对称矩阵, 则下列命题等价: (1)  $x^T A x$  是正定二次型(或  $A$  是正定矩阵); (2)  $A$  的正惯性指标为  $n$ ; (3) 存在可逆阵  $P$ , 使得  $A = P^T P$ ; (4)  $A$  的  $n$  个特征值  $\lambda_1, \lambda_2, \dots, \lambda_n$  全大于零.

(9) 设  $A, B$  都是  $n$  阶实对称矩阵, 且都正定, 那么  $AB$  是(C).

(A) 实对称矩阵      (B) 正定矩阵

(C) 可逆矩阵      (D) 正交矩阵

$A, B$  都是实对称矩阵, 因此  $(AB)^T = B^T A^T = BA$ , 但这不保证  $(AB)^T = AB$ , 即不保证  $AB$  是对称的. 故不选(A).

$AB$  不一定是对称的, 因此不一定是正定的.(按定义, 正定矩阵都是对称矩阵). 故不选(B).

(D)显然不可选.

$A, B$  都是正定矩阵, 因此都是可逆矩阵, 它们的乘积  $AB$  也是可逆矩阵. 选(C).

注意:

● 矩阵与行列式不要混淆. 矩阵是数表, 行列式是数. 矩阵用圆括号  $(A)$  或方括号  $[A]$ , 行列式用竖括号  $|A|$ .

● 矩阵与向量组的联系与区别

$$\blacksquare \quad A_{m \times n} = (\alpha_1, \alpha_2, \dots, \alpha_n) = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{pmatrix}$$

矩阵的秩  $R(A) =$  向量组的秩  $R\{\alpha_1, \alpha_2, \dots, \alpha_n\} = R\{\beta_1, \beta_2, \dots, \beta_m\}$ .

■ 矩阵  $A(m=n)$  是正交的, 即  $A^T A = E$  或  $A A^T = E$

$\Leftrightarrow A$  的列向量组是标准正交向量组, 即

$$[\alpha_i, \alpha_j] = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

$\Leftrightarrow A$  的行向量组是标准正交向量组, 即

$$[\beta_i, \beta_j] = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$



## 习题六

**6.1** 6.1 已知  $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \\ 3 & 4 & 3 \end{bmatrix}$ ,  $B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ ,  $n = 3$

(1) 在 Matlab 软件中输入  $A, B, n$

(2) 求  $A + B, \text{inv}(B); A^n; A * B$ .

(1)  $A = [1, 2, 3; 2, 2, 1; 3, 4, 3]$ ,  $B = [1, 1, 1; 0, 1, 1; 0, 0, 1]$ ,  $n = 3$

(2)  $A + B =$

$$\begin{bmatrix} 2 & 3 & 4 \\ 2 & 3 & 2 \\ 3 & 4 & 4 \end{bmatrix}$$

$\text{inv}(B) =$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$A^n =$

$$\begin{bmatrix} 92 & 120 & 102 \\ 66 & 86 & 72 \\ 138 & 180 & 152 \end{bmatrix}$$

$A * B =$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

**6.2** 6.2 已知  $a = (1, 2, 3, 4)$ ,  $b = (4, 3, 2, 1)$ ,  $n = 3$ , 求  $(a * b); a^n, \sqrt{a}$

>>  $a = [1 \ 2 \ 3 \ 4]; b = [4 \ 3 \ 2 \ 1]; n = 3;$

>>  $a * b =$

$$\begin{bmatrix} 4 & 6 & 6 & 4 \end{bmatrix}$$

>>  $a.^n =$

$$\begin{bmatrix} 1 & 8 & 27 & 64 \end{bmatrix}$$

>>  $\text{sqrt}(a) =$

$$\begin{bmatrix} 1.0000 & 1.4142 & 1.7321 & 2.0000 \end{bmatrix}$$

**6.3** 6.3 设  $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \\ 3 & 4 & 3 \end{bmatrix}$ ,  $B = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$ ,  $C = \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 3 & 1 \end{bmatrix}$

求矩阵  $X$  使其满足方程:  $AXB = C$ .

```
>> A=[1 2 3;2 2 1;3 4 3];B=[2 1;5 3];C=[1 3;2 0;3 1];
```

```
>> X=A\C/B
```

```
-2.0000    1.0000
 10.0000   -4.0000
-10.0000    4.0000
```

**6.4** 6.4 简述“;”在 Matlab 软件中的作用.

隔开矩阵的元素; 隔开函数的参数

**6.5** 6.5 在 Matlab 软件中建立 5 阶零矩阵, 单位矩阵, 全 1 矩阵, 随机矩阵.

```
>> zeros(5), eye(5), ones(5), rand(5, 5)
```

**6.6** 6.6 计算行列式 (1)  $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{vmatrix}$  (2)  $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 2 \\ 3 & -1 & -1 & 0 \\ 1 & 2 & 0 & -5 \end{vmatrix}$

```
(1) >> A=[1 1 1 1;1 -1 1 1;1 1 -1 1;1 1 1 -1];det(A)
```

```
ans =    -8
```

```
(2) >>A=[1 2 3 4;1 0 1 2;3 -1 -1 0;1 2 0 -5];det(A)
```

```
ans =   -24
```

**6.7** 6.7 求下列矩阵的秩, 迹和标准阶梯形.

(1)  $\begin{bmatrix} 1 & 5 & 10 & 0 \\ 7 & 8 & 18 & 4 \\ 17 & 18 & 40 & 10 \\ 3 & 7 & 17 & 1 \end{bmatrix}$  (2)  $\begin{bmatrix} 1 & 0 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 & 5 \\ 0 & 0 & 1 & 3 & 6 \\ 1 & 2 & 3 & 14 & 32 \\ 4 & 5 & 6 & 32 & 77 \end{bmatrix}$

```
(1) >> A=[1 5 10 0;7 8 18 4;17 18 40 10;3 7 17 1];
```

```
>> rank(A)
```

```
ans =    3
```

```
>> trace(A)
```

```
ans =    50
```

```
>> rref(A)
```

```
ans =    1.0000    0    0    0.7692
         0    1.0000    0    0
         0    0    1.0000   -0.0769
         0    0    0    0
```

```
(2) >> A=[1 0 0 1 4;0 1 0 2 5;0 0 1 3 6;1 2 3 14 32;4 5 6 32 77];
```

```
>> rank(A)
```

```
ans =    3
```

```
>> trace(A)
```

```
ans =    94
```

```
>> rref(A)
```

```
ans =    1    0    0    1    4
         0    1    0    2    5
```

$$\begin{array}{ccccc} 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}$$

6.8 6.8 已知方程组  $\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ 2x_1 + 3x_2 + x_3 + x_4 = 0 \\ 4x_1 + 5x_2 + 3x_3 + 3x_4 = 0 \end{cases}$  .求它的基础解系及通解.

```
>> A=[1 1 1 1;2 3 1 1;4 5 3 3];format rat;B=null(A,'r');syms k1 k2;
```

```
>> X=k1*B(:,1)+k2*B(:,2); pretty(X)
```

$$\begin{bmatrix} -2k_1 - 2k_2 \\ 0 \\ k_1 + k_2 \\ 0 \\ k_1 \\ 0 \\ k_2 \end{bmatrix}$$

6.9 6.9 求下列非齐次线性方程组  $\begin{cases} 2x_1 + 7x_2 + 3x_3 + x_4 = 6 \\ 3x_1 + 5x_2 + 2x_3 + 2x_4 = 4 \\ 9x_1 + 4x_2 + x_3 + 7x_4 = 2 \end{cases}$  的通解.

```
>> A=[2 7 3 1;3 5 2 2;9 4 1 7];b=[6 4 2]';
```

```
>> rank(A)
```

```
ans = 2
```

```
>> rank([A,b])
```

```
ans = 2
```

```
>> X=A\b
```

```
X =
```

```
-2/11
```

```
10/11
```

```
0
```

```
0
```

```
>> C=null(A,'r')
```

```
C =
```

```
1/11 -9/11
```

```
-5/11 1/11
```

```
1 0
```

```
0 1
```

$$\text{通解 } \mathbf{x} = \begin{pmatrix} -\frac{2}{11} \\ \frac{10}{11} \\ 0 \\ 0 \end{pmatrix} + k_1 \begin{pmatrix} \frac{1}{11} \\ -\frac{5}{11} \\ 1 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} -\frac{9}{11} \\ \frac{1}{11} \\ 0 \\ 1 \end{pmatrix} \quad (k_1, k_2 \in \mathbb{R}).$$

**6.10** 6.10 非齐次线性方程组 
$$\begin{cases} (2-a)x_1 + 2x_2 - 2x_3 = 1 \\ 2x_1 + (5-a)x_2 - 4x_3 = 2 \\ -2x_1 - 4x_2 + (5-a)x_3 = -a-1 \end{cases}$$

问当  $a$  为何值时, 此方程组有唯一解, 无解, 无穷多解? 并在有无穷多解时求通解.

```
>> syms a; A=[2-a 2 -2; 2 5-a -4; -2 -4 5-a]; b=[1 2 -a-1].';
>> det(A)
ans = 10-21*a+12*a^2-a^3
>> roots([-1 12 -21 10])
ans =
    10
     1      + 1/73058877i
     1      - 1/73058877i
```

当  $a \neq 10$  时, 系数行列式  $|A| \neq 0$ , 方程组有唯一解. 当  $a = 10$  时,

```
>> a=10; A=[2-a 2 -2; 2 5-a -4; -2 -4 5-a]; b=[1 2 -a-1].';
>> rank(A)
ans = 2
>> rank([A, b])
ans = 3
```

增广矩阵的秩与系数矩阵的秩不相等, 方程组无解.

**6.11** 6.11 求下列矩阵的特征值和特征向量.

$$(1) \begin{bmatrix} 5 & 6 & -3 \\ -1 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix} \quad (2) \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 3 & 6 \end{bmatrix}$$

(1)

```
>> A=[5 6 -3; -1 0 1; 1 2 1];
>> [B, C]=eigensys(A)
B =
[-2, 1]
[ 1, 0]
[ 0, 1]

C =
[ 2, 0, 0]
```

$$[0, 2, 0]$$

$$[0, 0, 2]$$

特征值  $\lambda_1 = \lambda_2 = \lambda_3 = 2$ , 特征向量  $k_1 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$  ( $k_1, k_2$  不全为零).

(2)

```
>> A=[1 2 3;2 1 3;3 3 6];
```

```
>> [B,C]=eigensys(A)
```

```
B =
```

$$[-1, -1, 1]$$

$$[-1, 1, 1]$$

$$[1, 0, 2]$$

```
C =
```

$$[0, 0, 0]$$

$$[0, -1, 0]$$

$$[0, 0, 9]$$

特征值  $\lambda_1 = 0$ , 特征向量  $k_1 \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$  ( $k_1 \neq 0$ );

特征值  $\lambda_2 = -1$ , 特征向量  $k_2 \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$  ( $k_2 \neq 0$ );

特征值  $\lambda_3 = 9$ , 特征向量  $k_3 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$  ( $k_3 \neq 0$ ).

**6.12** 6.12 求下列向量组的一组标准正交向量组.

$$\alpha_1 = (2, 1, 3, -1)^T, \alpha_2 = (7, 4, 3, -3)^T, \alpha_3 = (1, 1, -6, 0)^T, \alpha_4 = (5, 7, 7, 8)^T$$

```
>> a1=[2 1 3 -1]'; a2=[7 4 3 -3]'; a3=[1 1 -6 0]'; a4=[5 7 7 8]';
```

```
>> A=[a1, a2, a3, a4];
```

```
>> orth(A)
```

```
ans =
```

$$-452/913 \quad 1648/2843 \quad 1996/6857$$

$$-361/699 \quad 191/1795 \quad 519/1300$$

$$-608/1025 \quad -509/6716 \quad -648/809$$

$$-4855/13149 \quad -485/603 \quad 807/2387$$

与  $\alpha_1, \alpha_2, \alpha_3, \alpha_4$  等价的标准正交向量组只含3个向量, 这说明  $\alpha_1, \alpha_2, \alpha_3, \alpha_4$  是线性相关的. 事实上

```
>> null(A, 'r')
```

```
ans =
```

```
3
```

```
-1
```

```
1
```

```
0
```

方程组  $\mathbf{Ax} = \mathbf{0}$  有非零解  $\mathbf{x} = \begin{pmatrix} 3 \\ -1 \\ 1 \\ 0 \end{pmatrix}$ , 这意味着  $3\mathbf{a}_1 - \mathbf{a}_2 + \mathbf{a}_3 + 0\mathbf{a}_4 = \mathbf{0}$ .

**6.13** 6.13 用正交变换把下列二次型化为标准形, 并写出所作的正交变换.

$$f(x_1, x_2, x_3) = x_1^2 + 4x_2^2 + x_3^2 - 4x_1x_2 - 8x_1x_3 - 4x_2x_3$$

在编辑器中建立文件p7\_14.m

```
A=[1 -2 -4;-2 4 -2;-4 -2 1]
```

```
[U, T]=schur(A);
```

```
syms y1 y2 y3;
```

```
y=[y1; y2; y3];
```

```
X=vpa(U, 2)*y %取U的2位有效数字近似值
```

```
f=y.*T*y
```

运行后结果显示

```
>> p7_13
```

```
A =
```

```
1 -2 -4
```

```
-2 4 -2
```

```
-4 -2 1
```

```
X =
```

```
[ .67*y1-.46*y2-.59*y3]
```

```
[ .33*y1+.89*y2-.32*y3]
```

```
[ .67*y1+.17e-1*y2+.75*y3]
```

```
f = -4*y1^2+5*y2^2+5*y3^2
```

**6.14** 6.14 求正交矩阵  $\mathbf{T}$ , 使  $\mathbf{T}^{-1}\mathbf{AT}$  为对角矩阵.

$$(1) \begin{bmatrix} 3 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 1 \end{bmatrix} \quad (2) \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

(1)

```
>> A=[3 -2 0;-2 2 -2;0 -2 1];
```

```
>> format rat
```

```
>> [T, D]=schur(A)
```

```
T =
```

$$\begin{bmatrix} -1/3 & 2/3 & -2/3 \\ -2/3 & 1/3 & 2/3 \\ -2/3 & -2/3 & -1/3 \end{bmatrix}$$

```
D =
```

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$T = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \end{pmatrix} \text{是正交的且 } T^{-1}AT = \begin{pmatrix} -1 & & \\ & 2 & \\ & & 5 \end{pmatrix}.$$

(1)

```
>> A=[4 0 0;0 3 1;0 1 3];
```

```
>> [C, D]=schur(A); T=sym(C), D
```

```
T =
```

$$\begin{bmatrix} & 0, & 0, & 1] \\ [-\text{sqrt}(1/2), & \text{sqrt}(1/2), & & 0] \\ [ \text{sqrt}(1/2), & \text{sqrt}(1/2), & & 0] \end{bmatrix}$$

```
D =
```

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$T = \begin{pmatrix} 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \text{是正交的且 } T^{-1}AT = \begin{pmatrix} 2 & & \\ & 4 & \\ & & 4 \end{pmatrix}.$$

### 6.15 6.15 填空题

(1)在Matlab软件中，“;”的三个作用分别为 矩阵行间的分隔符；指令之间的分隔符；在赋值指令后阻止赋值结果在屏幕显示。

(2)在Matlab软件中，每条指令输入行结束后，必须按 回车键，该行才能执行。

(3)在Matlab软件中，矩阵的三种基本输入方式分别为 以直接列出元素的形式输入；利用MATLAB函数和语句创建数值矩阵；利用M文件创建和保存矩阵。

(4)在Matlab软件中，求行列式，特征值的指令分别为 det(A)；eig(A)。